

PROBLEMS FOR DOLLARS

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1. Determine all distinct positive integers a, b, c, d such that

$$\begin{aligned}a + b &= cd, \\ ab &= c + d.\end{aligned}$$

2. A convex polygon $P_0P_1 \dots P_n$ in the plane is divided with $n - 2$ diagonals into $n - 1$ triangles. Prove that the triangles can be labelled with $1, 2, \dots, n - 1$ in such a way that every point P_k is a vertex of the triangle labelled with k , for $k = 1, \dots, n - 1$.

3. For any real r , let $\lfloor r \rfloor$ be the largest integer not greater than r . Prove that for any positive integer n ,

$$\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor.$$

4. Odile and Evan play by randomly selecting a k -element subset $S \subseteq \{1, 2, \dots, 100\}$. If the sum of the elements in S is odd, then Odile wins, if it is even then Evan wins. Determine all k , $1 \leq k \leq 100$, such that Odile and Evan have the same chances of winning.

5. Is there a point set T in the (3-dimensional Euclidean) space such that every plane contains at least one point of T but no plane contains infinitely many points of T ?

6. Determine the largest h and the smallest H such that

$$h < \frac{a_1}{a_1 + a_2} + \frac{a_2}{a_2 + a_3} + \dots + \frac{a_{49}}{a_{49} + a_{50}} + \frac{a_{50}}{a_{50} + a_1} < H$$

holds for every positive numbers a_1, a_2, \dots, a_{50} .