

**DE FINAL - EXAM SUMMER 2003**

1. Find the general solution of  $y' + \frac{2}{x}y - 2y^2 = 0$ .
2. Find the general solution of  $y'' - y = \frac{e^{2x}}{1 + e^x}$ .
3. Consider the ODE  $(1-x^2)y'' + 5xy' - 2y = 0$ .  $x = 0$  is an ordinary point and consequently we can expect solutions of the form  $\sum_{n=0}^{\infty} c_n x^n$  which converge in the interval  $(-1, 1)$ . Find the recursion relation for the  $c'_n s$ .
4. Consider the ODE  $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ . We assume that there exists a series solution of the form  $\sum_{n=0}^{\infty} c_n x^n$ . Given that  $y(0) = 1$ ,  $y'(0) = 2$ , and

$$(n + 1)(n + 2)c_{n+2} + (8 - 2n)c_n = 0, \text{ for } n = 0, 1, 2, \dots$$

write out the first four terms of the solution.

5. Find  $\mathcal{L}(f(t))$  when

$$f(t) = 2 \text{ for } 0 \leq t \leq 2 \text{ and } f(t) = \sin 3(t - 2) \text{ for } t \geq 2.$$

6. Find the inverse Laplace transform  $\mathcal{L}^{-1}(F(s))$  if

$$(a) F(s) = \frac{(2s + 3)}{(s - 3)^2 + 25}.$$

$$(b) F(s) = e^{-3s} \frac{s + 1}{s^2 + 4s + 53}$$

7. Consider the IVP

$$y''' + 3y'' - 2y' + 6y = 1 - \mathcal{U}(t - 1), \quad y(0) = 0, y'(0) = 1, \text{ and } y''(0) = 0.$$

Find the Laplace transform of the solution, i.e.  $\mathcal{L}(y(t))$ .

8. Write the solution of the following IVP as a convolution integral:

$$y'' + y = e^t \cos 2t, \quad y(0) = y'(0) = 0.$$

9. (a) Find the eigenvalues of the matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 2 & 2 \end{pmatrix}$$

- (b) Given that the eigenvalues of the following matrix are

$\lambda_1 = 1, \lambda_2 = 0$ , and  $\lambda_3 = 2$ , find the corresponding eigenvectors.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

10. Write the first six terms of the unique solution to the IVP:

$y'' - xy' + y = 0$ ,  $y(1) = 1$ ,  $y'(1) = 2$ . **Hint: Compute  $y'''$ ,  $y''''$ , etc.** by differentiating the DE and substituting  $x = 1$ . The solution can be written in the form:  $y(x) =$

$$\sum_{n=0}^{\infty} a_n(x-1)^n.$$

11. Find the unique solution of the following initial value problem:  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{t})$  where

$$\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix},$$

$$\mathbf{F}(\mathbf{t}) = \begin{pmatrix} e^t \\ 2 \end{pmatrix}$$

and

$$\mathbf{x}(\mathbf{0}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$