ADJACENT VERTEX DISTINGUISHING EDGE-COLORINGS

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ABSTRACT. An adjacent vertex distinguishing edge-coloring of a simple graph G is a proper edge-coloring of G such that no pair of adjacent vertices meets the same set of colors. The minimum number of colors $\chi'_a(G)$ required to give G an adjacent vertex distinguishing coloring is studied for graphs with no isolated edge. We prove $\chi'_a(G) \leq 5$ for such graphs with maximum degree $\Delta(G) = 3$ and $\chi'_a(G) \leq \Delta(G) + 2$ for bipartite graphs. These bounds are tight. For k-chromatic graphs G without isolated edges we prove a weaker result of the form $\chi'_a(G) = \Delta(G) + O(\log k)$.

1. INTRODUCTION

Let G be a simple graph. We say a proper edge-coloring of G is *adjacent vertex distinguishing* or an *avd*-coloring if for any pair of adjacent vertices x and y, the set of colors incident to x is not equal to the set of colors incident to y. It is clear that an avd-coloring exists provided G contains no isolated edge. A *k-avd-coloring* is an avd-coloring using at most k colors. Let $\chi'_a(G)$ be the minimum number of colors in an avd-coloring of G. In [6] the following conjecture was made.

Conjecture 1. *If G is a simple connected graph on at least 3 vertices and $G \neq C_5$ (a 5-cycle) then $\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2$.*

Since $\chi'_a(G)$ is at least as large as the edge-chromatic number of G it is clear that $\chi'_a(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum degree of any vertex in G. There are many examples of graphs for which $\chi'_a(G) > \Delta(G) + 1$. For example, consider a graph of the form $G = K_{n,n} - H$ where H is a 2-factor of the complete bipartite graph $K_{n,n}$ containing no C_4. Assume we have an avd-coloring of G using $\Delta(G) + 1$ colors. Then each vertex is not incident to precisely one color, and assigning this missing color to each vertex gives a proper vertex-coloring of G with $\Delta(G) + 1$ colors. Since G is bipartite with equal class sizes, each color must occur the same number of times on the vertices of each class. Since $\Delta(G) + 1 = n - 1$ there is a color that occurs twice in each class, but the vertices with this color do not form an independent set in G. Hence $\chi'_a(G) > \Delta(G) + 1$.

We shall give the following upper bounds for $\chi'_a(G)$.

Date: September 5, 2002.

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Theorem 1.1. *If G is a graph with no isolated edges and $\Delta(G) = 3$ then $\chi'_a(G) \leq 5$.*

Theorem 1.2. *If G is a bipartite graph with no isolated edges then $\chi'_a(G) \leq \Delta(G) + 2$.*

Theorem 1.3. *If G is a k-chromatic graph with no isolated edges then $\chi'_a(G) \leq \Delta(G) + O(\log k)$.*

In particular, Conjecture 1 holds for all bipartite graphs and all graphs with $\Delta(G) \leq 3$. Note that even for bipartite graphs, Conjecture 1 is best possible, as the example above shows. Theorem 1.1 will be proved in Section 2, Theorem 1.2 will be proved in Section 3, and Theorem 1.3 will be proved in Section 4.

Adjacent vertex distinguishing colorings are related to vertex distinguishing colorings in which *every* pair of vertices sees distinct color sets. This concept has been studied in many papers, see for example [1–5].

2. Graphs with $\Delta(G) = 3$.

We start with the special case of regular graphs having a hamiltonian cycle. Our coloring scheme is based on the idea of using the four elements of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ to color the hamiltonian cycle, defining the colors used algebraically, and a new fifth color for the chords forming a 1-factor. Local adjustments will be used to complete the colorings.

Lemma 2.1. *If G is a 3-regular hamiltonian graph then G has a 5-avd-coloring.*

Proof. Let the five colors be the elements $\{0, a, b, c\}$ of the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ together with the extra color 5. We have addition defined on K by $x + x = 0$ for all x and $a+b = c$. Let $C = x_1 \dots x_n$ be a hamiltonian cycle of G and let I be the remaining 1-factor of G. We may assume $G \neq K_4$ (see Figure 1 for a 5-avd-coloring of K_4), so by Brooks' theorem G has a vertex 3-coloring $f \colon V(G) \to \{a, b, c\}$. We may also assume that each of the three colors occurs at least once on G, otherwise a single vertex can be recolored to introduce the missing color. Let $S = \sum_{i=1}^{n} f(x_i) \in K$.

If $S = 0$ then label $x_n x_1$ with 0 and inductively label $x_i x_{i+1}$ for $i = 1, \dots, n-1$ so that $f(x_i)$ is the sum (in the group K) of the colors on $x_{i-1}x_i$ and $x_i x_{i+1}$. Equivalently, the color on $x_i x_{i+1}$ is the sum of the color on $x_{i-1}x_i$ and $f(x_i)$. Then $f(x_n)$ is the sum of the colors on $x_n x_1$ and $x_{n-1}x_n$. Color the 1-factor I with color 5. Each vertex x sees color 5 and two colors from K summing to $f(x)$. Since $f(x) \neq 0$ these two colors from K are distinct, and since $f(x) \neq f(y)$ for any two adjacent vertices x and y, the color sets at x and y must be distinct. Thus the coloring is a 5-avd-coloring of G.

Now suppose $S \neq 0$. Without loss of generality we may assume $S = c$. Pick any vertex x_i with $f(x_i) = c$. Let $x_i x_j \in I$. Then $f(x_j)$ is either a or b. Recolor x_j with b or a, respectively. Now $S = 0$ and we can recolor the edges of the hamiltonian cycle as above (see Figure 1). Coloring I with 5 gives a proper edge-coloring that distinguishes adjacent vertices except possibly at x_j. Since $f(x_i) \neq f(x_j)$ the pair of colors from K meeting x_i cannot be disjoint from the pair that meet x_j. Hence there must be some color of K missing from the edges incident to x_i or x_j. Recoloring the edge $x_i x_j$ with this missing color gives a 5-avd-coloring of G. The vertices x_i and x_j are distinguished from each other

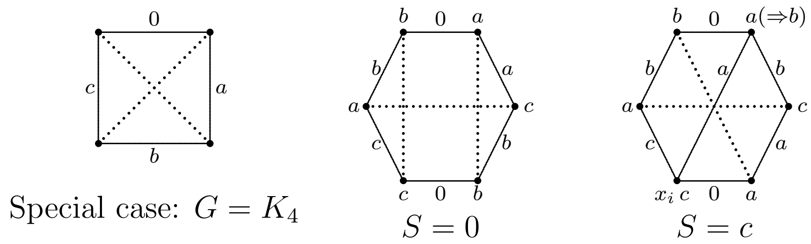

FIGURE 1. Colorings in Lemma 2.1. Dotted edges are colored 5.

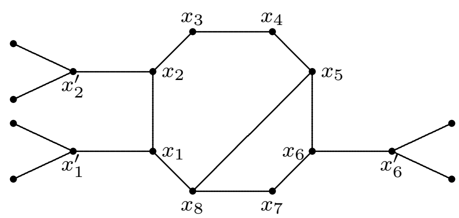

FIGURE 2. Graph H with $V_Y = \{x_1, x_2, x_6\}$, $V_C = \{x_5, x_8\}$, $V_S = \{x_3, x_4, x_7\}$.

since $f(x_i) \neq f(x_j)$ and are distinguished from all other vertices since all other vertices meet color 5. □

We shall now assume that G is 3-regular with a 1-factor, but is not necessarily hamiltonian. Since G has a 1-factor, G can be written as a union of this 1-factor and a collection of cycles. We shall show that under certain conditions we can extend a partial coloring to each cycle in turn.

We shall first find suitable colorings of graphs H of the following form. Let H be a cycle $C = x_1 \ldots x_n$ with some extra 3-stars and chords added. To be precise, partition $V(C)$ as $V_Y \cup V_C \cup V_S$. For $x_i \in V_Y$, H will contain an edge $x_i x_i'$, $x_i' \notin V(C)$, where x_i' is joined to two degree 1 vertices. For $x_i \in V_C$, H will contain a chord $x_i x_j$ where $x_j \in V_C$. For $x_i \in V_S$, $d_H(x_i) = 2$ (see Figure 2).

We shall color such graphs so that adjacent degree 3 vertices are distinguished. We shall specify the colors incident to the x_i' for all $x_i \in V_Y$, and try to extend the coloring to the rest of H.

Lemma 2.2. *Let H be a graph as above with $|V_S| \geq 2$. Suppose the edges incident to each x_i' with $x_i \in V_Y$ are properly colored by $K \cup \{5\}$ with $x_i x_i'$ colored 5. Then we can properly color the remaining edges of H with colors from $K \cup \{5\}$ so that adjacent degree 3 vertices are distinguished. Moreover, if $x_i \in V_S$ then either x_i meets color 5, or both neighbors of x_i meet color 5.*

Proof. We partition V_S into two sets V_I and V_M as follows. If $x, y \in V_S$ are adjacent on C, color the edge xy with color 5 and place x, y in the set V_M. Repeat with other adjacent pairs of V_S (that have not been used already) until $V_I = V_S \setminus V_M$ is an independent set. We shall now 3-color the degree 3 vertices of H with $\{a, b, c\}$. For $x_i \in V_Y$, set the color of x_i' to be the sum of the two colors of K incident to x_i'. Extend this vertex-coloring to a proper vertex-coloring of $V(H) \setminus V_S$ using a greedy algorithm. Proceed around C, starting at any vertex immediately after a vertex of V_S, coloring each vertex of $V_Y \cup V_C$ in turn with any color from $\{a, b, c\}$ that ensures that the coloring is still proper.

If $V_M = \emptyset$ then $|V_I| \geq 2$. Coloring vertices of V_I (not necessarily properly) with colors from $\{a, b, c\}$ we can ensure that the sum of the vertex colors on C is $0 \in K$. If $V_M \neq \emptyset$, color V_I arbitrarily with $\{a, b, c\}$. Color the uncolored edges round C as in Lemma 2.1. At each vertex we add the vertex color in the Klein group to get the color of the next edge. The edge after any pair of vertices from V_M can be colored arbitrarily with any color from K. Color each chord of C with color 5. The resulting coloring satisfies the conditions of the Lemma. □

Note that if we add an edge $x_i x_i'$ to H for some $x_i \in V_S$ and if x_i meets color 5 in a coloring given by Lemma 2.2, then $x_i x_i'$ can be colored with some element of K so as to ensure the coloring is still avd. This is because there are 3 colors which make the coloring proper and at most two of these will fail to distinguish x_i from x_{i+1} or x_{i-1}. If x_i does not meet color 5 then $x_i x_i'$ may be colored with either remaining color of K since both x_{i+1} and x_{i-1} meet color 5.

Lemma 2.3. *Let H be a graph as above with $V_S = \emptyset$ and $x_1 \in V_Y$. Suppose the edges incident to each x_i' with $x_i \in V_Y \setminus \{x_1\}$ are properly colored by $K \cup \{5\}$ with $x_i x_i'$ colored 5 and, either (a) the edges incident to x_1' are colored with one of them other than $x_1 x_1'$ colored 5, or (b) the colors incident to x_1' are colored except for $x_1 x_1'$ which remains uncolored. Then the coloring can be completed to form a 5-avd-coloring of H using $K \cup \{5\}$. Moreover, in this coloring, either x_1 meets color 5, or both x_2 and x_n meet color 5.*

Proof. We shall provisionally color all chords $x_i x_j$ of C with color 5. As in the proof of Lemma 2.1 we shall 3-color the vertices of H with $\{a, b, c\}$. Each x_i' for $x_i \in V_Y$ is assigned the sum of the colors of K meeting it in H. We can 3-color the vertices $x_2, \ldots, x_n$ so that the coloring is proper using a greedy algorithm. The vertex x_1 will remain uncolored. Let this coloring be denoted by f and write $S = \sum_{i=2}^{n} f(x_i)$. If $S \neq 0$ then assign $x_1 x_2$ any color of K, and color the edges round the cycle as in the proof of Lemma 2.1. This gives four possible avd-colorings of $H - x_1'$ depending on the choice of color for $x_1 x_2$, and yields either $\{0, S\}$ or $K \setminus \{0, S\}$ as the pair of colors on $x_n x_1$ and $x_1 x_2$.

Assume that there is a chord $x_i x_j$ of C which does not meet either x_2 or x_n. Suppose without loss of generality that $f(x_i) = a$ and $f(x_j) = b$. Recolor either x_i or x_j with c and change color 5 of $x_i x_j$ to some available color of K as in the proof of Lemma 2.1. In this way we can construct colorings with three distinct values of S (the original coloring, the coloring changing $f(x_i)$, and the coloring changing $f(x_j)$). At least two of these will have $S \neq 0$, and we obtain colorings with four possible values for the pair of colors on $x_n x_1$ and $x_1 x_2$. These four pairs form the edges of a C_4 inside K_K — the complete graph on the color set K. Moreover, both x_2 and x_n meet color 5, so are distinguished from x_1 regardless of the color (in K) of $x_1 x_1'$. In case (a) we are done since we can choose a coloring for which the pair of colors on $x_n x_1$ and $x_1 x_2$ avoids the color of $x_1 x_1'$. In case (b) we are done since we can choose a coloring for which the pair of colors on $x_n x_1$ and $x_1 x_2$ is neither equal nor

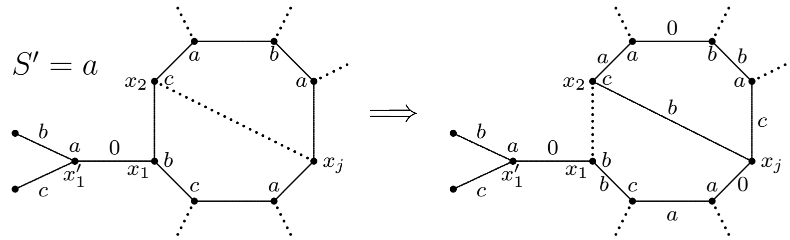

FIGURE 3. Case when x_2x_j is a chord of H.

disjoint from the pair that meet x_1'. Then there is at least one remaining color of K to color x_1x_1'.

Assume now that x_2x_j is a chord of C. In case (b) color x_1x_1' arbitrarily with some color of K so that the coloring is proper at x_1'. Restart from scratch and three-color the vertices of H with $\{a, b, c\}$ as follows. As before, x_i' gets the sum of the colors of K meeting it when $x_i \in V_Y$. For the cycle C, start the coloring at x_{j-1} working backwards greedily until we reach x_2. The vertex x_2 can be colored in two ways. We pick one that ensures that $S' = \sum_{i=2}^{j-1} f(x_i) \neq 0$. Now continue coloring greedily with x_1, and then $x_n, \ldots, x_{j+1}$. The vertex x_j will remain uncolored.

Starting at x_1 and working backwards round C color the edges so that $f(x_i)$ is the sum of the colors of K meeting x_i. For the edge x_jx_{j-1} pick either color of K that is not the same as the color of $x_{j+1}x_j$, or the sum of this color and S'. Continue coloring the edges of C as in Lemma 2.1 until we get to x_2. Color x_1x_2 with color 5 and color x_2x_j with the sum of $f(x_2)$ and the color on x_2x_3 (see Figure 3). This color will be the sum of S' and the color on x_jx_{j-1}, so is distinct from the colors on $x_{j+1}x_j$ and x_jx_{j-1}. The resulting coloring satisfies the conditions of the Lemma.

We may now assume there are no chords, so $V_C = \emptyset$. Restart by coloring the vertices of $C - x_1$ with $\{a, b, c\}$ as follows. Assume $f(x_3') = \cdots = f(x_j') = a \neq f(x_{j+1}')$ (or $j = n$). Color x_{j+1} with a and greedily color x_i for $i > j + 1$. The vertices $x_3,\ldots,x_j$ can be colored alternately by b and c, starting with either b or c. The vertex x_2 will be colored a, b, or c (possibly equal to the color of x_2', but not equal to the color of x_3). Let $S = \sum_{i=2}^{n} f(x_i)$.

We now list the possible colorings. For each choice of colorings of x_2 and x_3, there are four possible values of S depending on the value of $S' = \sum_{i=j+1}^{n} f(x_i)$ and j. The following table lists the possible values of S.

$f(x_2)$	$f(x_3)$	S (j even)				S (j odd)			
a	c	a	0	c	b	b	c	0	a
a	b	a	0	c	b	c	b	a	0
b	c	b	c	0	a	a	0	c	b
c	b	c	b	a	0	a	0	c	b

Each value of S' and j gives a column in the table for S. Since $f(x_2)$ and $f(x_3)$ can be changed independently of S' we have several choices for the vertex coloring for each

S' and j. We describe several cases in which we can find a suitable corresponding edge coloring.

Case A. $f(x_2) \neq f(x_2')$, $S \neq 0$.
As in Lemma 2.1, we can edge color C starting at x_1x_2. Since $S \neq 0$, the colors on x_1x_n and x_1x_2 will be distinct. Depending on the choice of x_1x_2, the pair of colors meeting x_1 can be chosen to be either $\{0, S\}$ or $K \setminus \{0, S\}$. (We assume x_1x_1' is uncolored for now).

Case B. $f(x_2) = f(x_2') \neq S$, $S \neq 0$.
As before we color the edges of C. However, this time only two choices for x_1x_2 are allowed since we must ensure that x_2 is distinguished from x_2' (either color not meeting x_2' will do for x_1x_2). These choices differ by the addition of $f(x_2)$ to every edge of C and since $f(x_2) \notin \{0, S\}$, this swaps the pairs of colors $\{0, S\}$ and $K \setminus \{0, S\}$ on x_nx_1 and x_1x_2.

Case C. $f(x_2) = f(x_2') = S$, $S \neq 0$.
Unfortunately, both choices above of the color for x_1x_2 give the same pair of colors on x_nx_1 and x_1x_2. Hence we can only guarantee colorings exist making x_1 meet *one* of the pairs $\{0, S\}$ or $K \setminus \{0, S\}$.

For each S' and j (corresponding to a column in the table above) there are always at least two possible non-zero values for S. Moreover, for any choice of $f(x_2')$, we can find two choices of $f(x_2)$ and $f(x_3)$ with distinct values of $S \neq 0$, at least one of which either has $f(x_2) \neq f(x_2')$ or has $f(x_2) = f(x_2') \neq S$. Hence the set of pairs of colors meeting x_1 can be chosen as any edge of a path of edge length 3 in K_K (one value of S gives a matching in K_K, the other at least one more edge in K_K).

In case (a) we are now done, since there is always a choice of the pair of colors that does not include the color on x_1x_1'. Also, x_1 and x_1' are distinguished since only x_1' meets color 5. In case (b) there is some choice for this pair of colors that is not equal or disjoint from the pair of colors meeting x_1'. Hence there is a choice of color in K for x_1x_1' which makes the coloring proper and distinguishes x_1 and x_1'. □

Theorem 2.4. *If G is a 3-regular graph containing a 1-factor, then there exists a 5-avd-coloring of G.*

Proof. Without loss of generality we may assume G is connected. Decompose G as a 1-factor I and a union of cycles C_i. If there is only one cycle then G is hamiltonian and we are done by Lemma 2.1. Otherwise construct a new graph M with vertex set $V(M)$ equal to the set of cycles C_i and edges joining C_i and C_j when there is an edge of I joining some vertex of C_i to some vertex of C_j. Since G is connected, M is also connected. Pick a spanning tree T of M. Decompose T as a vertex disjoint union of stars S_j, $|V(S_j)| \geq 2$. For each S_j let G_j be the subgraph of G with edge set made up from the edges of the cycles C_i of S_j, together with their chords in G and one edge of I joining C_i and $C_{i'}$ for each edge $C_iC_{i'}$ of S_j. Color G in the following way. Each edge (of I) that does not lie in any G_j will be colored 5. Now color each G_j in turn. If the star S_j has at least 3 vertices in M, use Lemma 2.2 to color the central cycle C_{i_0} of S_j. The graph H of Lemma 2.2 consists of C_{i_0}, its chords in G, and some 3-stars. The vertices of C_{i_0} incident to an edge joining C_{i_0} to

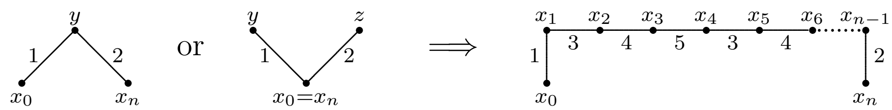

FIGURE 4. Case when G contains adjacent degree 2 vertices

another cycle in G_j will be placed in V_S and we attach a 3-star to each remaining vertex of C_{i_0} that does not meet a chord of C_{i_0}. The edges of this 3-star correspond in an obvious way with some of the edges of G (although the degree 1 vertices of H may not necessarily be distinct in G). We color the edges of the 3-stars with the corresponding colors already assigned in G, or arbitrarily (but properly) if no color has been assigned yet. Note that the edge of a 3-star incident to C_{i_0} will be colored 5. Lemma 2.2 now extends the coloring to the edges and chords of C_{i_0}. Now color the edges x_iy_i joining C_{i_0} to the other cycles C_i of G_j with some color of K if x_i meets color 5 on C in such a way that the coloring is avd on C (see note after Lemma 2.2). Otherwise leave x_iy_i uncolored. We now color the other cycles C_i of G_j using Lemma 2.3 in a similar manner using the edge x_iy_i as the edge $x_1x'_1$ of Lemma 2.3. The conditions of Lemma 2.3 ensure that we can find a coloring that is 5-avd regardless of the choices of colors on the edges already colored. If the star S_j consists of just two vertices, use Lemma 2.3 on both constituent cycles. For the first cycle we use case (b) of Lemma 2.3. This will result in the edge $x_1x'_1$ between the cycles being colored. If x_1 does not meet color 5 then uncolor $x_1x'_1$. Now color the other cycle using case (a) or (b) of Lemma 2.3. If $x_1x'_1$ is recolored with a new color then x_1 does not meet color 5 but both its neighbors on the first cycle do. Hence the coloring is still avd on the first cycle. □

Proof of Theorem 1.1.
We shall prove Theorem 1.1 by induction on $|E(G)|$. Paths and cycles on at least 3 vertices have 5-avd-colorings, so we may assume that G is connected with maximum degree 3.

Assume x is a vertex of degree 1 in G. Let y be the neighbor of x. Then y is of degree 2 or 3. Since $G \neq P_3$ we can find a 5-avd-coloring of $G' = G - x$ by induction. In G' y has degree at most 2, so there are at least three colors not incident to y. At most two of these colors cannot be used to color xy as they may result in y meeting the same set of colors as some neighbor in G'. However there is still at least one color that can be given to xy so that the coloring is avd. Hence we may assume G contains no degree 1 vertex.

Assume two vertices of degree 2 are adjacent in G. Let $x_0x_1x_2 \dots x_n$, $n > 2$, be a *suspended trail* in G, i.e., a trail with $d_G(x_0) = d_G(x_n) = 3$ and $d_G(x_i) = 2$ for $0 < i < n$. If $x_0 \neq x_n$ let G' be the graph obtained by contracting this path to x_0yx_n. If $x_0 = x_n$ let G' be the graph obtained by deleting the vertices $x_1, \dots, x_{n-1}$ and adding two degree one vertices y, z to $x_0 = x_n$ (see Figure 4). By induction G' has a 5-avd-coloring. We may assume without loss of generality that the edge x_0y has color 1 and x_ny (or x_nz) has color 2. The edges x_ix_{i+1} of G can be colored with 1 for $i = 0$, 2 for $i = n-1$, and cyclically with the colors $\{3, 4, 5\}$ for other values of i.

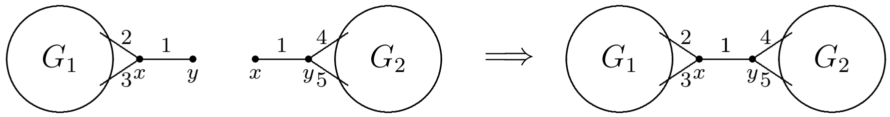

Figure 5. Case when G contains a bridge

Hence we can assume that any vertex of degree 2 is adjacent only to vertices of degree 3. If G contains a bridge xy, let G_1 and G_2 be components of $G - xy$ with $x \in V(G_1)$ and $y \in V(G_2)$. Give $G_1 \cup xy$ and $G_2 \cup xy$ 5-avd-colorings by induction. By permuting the colors on $G_2 \cup xy$, we can assume the edge xy receives the same color in each coloring and the color set incident to x in $G_1 \cup xy$ is not the same as the color set incident to y in $G_2 \cup xy$. This now gives a 5-avd-coloring of G (see Figure 5).

Hence we can assume that G is a graph with maximum degree 3, no vertices of degree 1, no pair of adjacent degree 2 vertices, and bridgeless. A cubic graph without a 1-factor must have at least three bridges, so if G contains no degree 2 vertices we are done. If G contains degree 2 vertices, then let G' be the graph obtained by taking two copies of G and joining their corresponding degree 2 vertices by an edge. Then G' is 3-regular and contains at most one bridge. Hence G' has a 1-factor and so by Theorem 2.4 G' has a 5-avd-coloring. This coloring of G' induces a 5-avd-coloring of G since no two vertices of degree 2 are adjacent in G. □

3. Bipartite graphs

If G has an edge coloring with colors $c_1, \dots, c_k$, write $G\{c_1, \dots, c_r\}$ for the subgraph of G consisting of the edges colored with a color in $\{c_1, \dots, c_r\}$. Write $S(v)$ for the set of colors incident to v. Write $\chi'(G)$ for the edge-chromatic number of G.

The bound $\chi'_a(G) \leq \Delta(G) + 3$ for regular bipartite graphs comes rather easily using the 1-factorization of regular bipartite graphs. To see this observe that a 2-regular bipartite graph H with bipartition $V(H) = A \cup B$ has a straightforward 5-avd-coloring along each cycle such that $S(a) \in \{\{1,2\}, \{3,4\}, \{3,5\}\}$ and $S(b) \in \{\{1,4\}, \{2,3\}, \{4,5\}, \{1,3\}\}$ for every $a \in A$ and $b \in B$. For $\Delta(G) > 2$ use this coloring for a 2-factor $H \subseteq G$ and give $G \setminus H$ any proper coloring with the remaining $\Delta(G) - 2$ colors. To obtain the bound $\Delta(G) + 2$ for any bipartite graph, much more effort will be required.

Lemma 3.1. *If G is a bipartite graph with no isolated edges, then there exists a proper edge-coloring with colors $\{1, \dots, \chi'(G)\}$ such that*

- A *if $uv \in E(G) \setminus E(G\{1,2\})$ then either $\{1,2\} \subseteq S(u)$ or $\{1,2\} \subseteq S(v)$.*
- B *if C is a cycle in $G\{1,2\}$ which does not meet color 3 in G then $\{1,2,3\} \subseteq S(v)$ for every neighbor v in $G \setminus C$ of any vertex of C.*
- C *if C is a cycle in $G\{1,2\}$ which does meet color 3 in G, then there exists a $u \in V(C)$ and $uv \in E(G\{3\})$ with $\{1,2\} \subseteq S(v)$.*
- D *if uv is an isolated edge in $G\{1,2,3\}$ then $S(u) \neq S(v)$.*

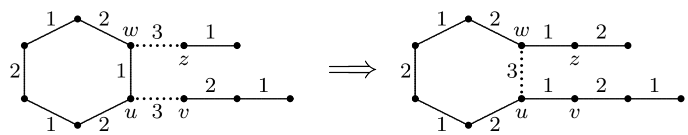

FIGURE 6. Proof of Condition C

Proof. Consider the set of edge-colorings of G with $\chi'(G)$ colors. For all such colorings pick one such that

(1) $G\{1,2\}$ has maximal edge count,
(2) subject to (1), $G\{1,2\}$ has the minimum number of components,
(3) subject to (1–2), $G\{3\}$ has maximal edge count,
(4) subject to (1–3), the number of edges uv in G failing condition D is minimal.

We shall show that such a coloring satisfies conditions A–D.

Condition A. Let $uv \in E(G) \setminus E(G\{1,2\})$ be an edge with $\{1,2\} \not\subseteq S(u), S(v)$. Then u and v are either isolated vertices or the end-vertices of paths in $G\{1,2\}$. By recoloring uv with either color 1 or 2 (and possibly interchanging colors 1 and 2 in the component of v in $G\{1,2\}$) we obtain a proper edge coloring with more edges colored $\{1,2\}$ contradicting (1). Note that if u and v are endvertices of the same path in $G\{1,2\}$ then since G is bipartite, the edge uv can be recolored without changing any colors on this path.

Condition B. Assume $uv \in E(G)$ with $u \in V(C)$, $v \notin V(C)$, and $\{1,2,3\} \not\subseteq S(v)$. Note that uv is not colored with 1, 2, or 3. If $3 \notin S(v)$ then we can recolor uv with 3, contradicting (3). Hence without loss of generality $1 \notin S(v)$. Recolor uv with 1 and recolor the color 1 edge on C meeting u with color 3. This contradicts condition (2).

Condition C. Suppose $u \in V(C)$ meets color 3 on an edge uv with $1 \notin S(v)$. Clearly $v \notin V(C)$. Let w be the neighbor of u on C with uw colored 1. If $3 \notin S(w)$, then recolor uw with 3 and uv with 1. This gives a coloring contradicting (2). If $3 \in S(w)$, let zw be the edge incident to w colored 3. If $\{1,2\} \subseteq S(z)$ then we are done, otherwise we can assume z is either an isolated vertex or the end of a path in $G\{1,2\}$. Recolor uv and wz with 1, uw with 3, and, if necessary, swap colors 1 and 2 on the path from z in $G\{1,2\}$ so as to make the coloring proper (see Figure 6). If the paths in $G\{1,2\}$ meeting v and z are the same then recoloring this path will be unnecessary since G is bipartite. We now have a new coloring with more edges in $G\{1,2\}$ contradicting (1).

Condition D. Let u_1v_1 be an isolated edge of $G\{1,2,3\}$. By Condition A, u_1v_1 is colored with either 1 or 2. Since G contains no isolated edge we can assume $d_G(u_1) \geq 2$ and u_1 meets another color $k > 3$ on some edge of G. Swap colors 3 and k along a Kempe chain (component path of $G\{3,k\}$) starting at u_1 in G. This will reduce the number of edges failing condition D unless the other end-vertex v_2 of this chain lies in some isolated edge u_2v_2 of $G\{1,2,3\}$ and after the recoloring $3 \notin S(u_2) = S(v_2)$. In this case u_2 also meets color k, so we can form a new Kempe chain starting at u_2 using colors 3 and k. Repeating this process we get a sequence of Kempe chains on colors 3 and k from u_i to v_{i+1}. Note that properties (1)–(3) still hold after these recolorings. Eventually this process must terminate with a coloring reducing the number of edges failing condition D, or with some $v_r = v_1$. However in this last case recoloring all these Kempe chains reduces the number of edges colored 3, contradicting (3). □

Proof of Theorem 1.2.
Color G as in Lemma 3.1. The edges of $G\{1,2\}$ form a set of vertex disjoint paths and even cycles. Construct a new graph M with vertex set equal to the non-singleton components C_i of $G\{1,2\}$ and edges joining C_i and C_j when either

1. there is an edge of $G\{3\}$ joining a vertex of degree 2 in C_i to a vertex of degree 2 in C_j; or
2. either C_i or C_j is a single edge and there is an edge of $G\{3\}$ joining a vertex of C_i to a vertex of C_j.

As in the proof of Theorem 2.4, we take a star decomposition $\{S_j\}$ of a spanning forest of M and consider a corresponding subgraph G' of $G\{1,2,3\}$ in G consisting of the induced subgraphs in $G\{1,2,3\}$ of each cycle C_i and a choice of edges from $G\{3\}$ joining C_i and $C_{i'}$ when $C_iC_{i'}$ is an edge of one of the stars in the star decomposition. Note that the graph M may contain isolated vertices so some of the stars may be isolated vertices as well. We shall color every edge that does not lie in G' with its original color in G. The colors 1 and 2 do not appear on these edges. The subgraph G' will be colored with colors from $K \cup \{3\}$ where $K = \{0, a, b, c\}$ is the Klein group, so as to obtain an avd-coloring of G using at most two more colors.

We say a component C_i of $G\{1,2\}$ is *bad* if it is either a single edge where the endvertices are not distinguished in the coloring of G, or a cycle of length congruent to 2 mod 4 that meets color 3, but has no color 3 chord. All other C_i will be called *good*.

Consider the structure of a component of G'. If a component C_i of $G\{1,2\}$ is isolated in G', and C_i is a cycle, then by conditions B and C we see that either C_i has a color 3 chord, or C_i does not meet color 3, but all its neighbors do. If C_i is a path, then by conditions A and D it has at least three vertices. In all cases C_i is good.

Now we consider the stars S_j. Suppose we have a star with central component C_0 and end-components $C_1, \dots, C_r$. If $r > 2$ delete the edge from C_0 to good components C_i in S_j, $i > 0$, until either $r = 1$ or all C_i, $i > 0$, are bad. If $r = 1$ and C_0 and C_1 are good, delete the edge joining them in S_j. If C_0 is bad and C_1 is good, we consider C_1 to be the center of the star. Furthermore, if C_0 is an edge, then C_1 is not an edge (otherwise we would have two adjacent vertices of degree 1 in $G\{1,2\}$ contradicting condition A). In this case also we swap C_0 and C_1, so we can assume without loss of generality that C_0 is not a single edge when $r = 1$ (or $r > 2$).

Hence we may assume each star S_j is either an isolated good C_i or a star with all end-components bad. In the second case, the color 3 edges in G' joining C_0 to the end-components are incident to degree 2 vertices of C_0 except in the case when $r = 2$ and C_0 is a single edge.

We now recolor G' with colors from the Klein group $K = \{0, a, b, c\}$. Let G have bipartition $V(G) = A \cup B$. We shall provisionally color the vertices of A with $a \in K$ and

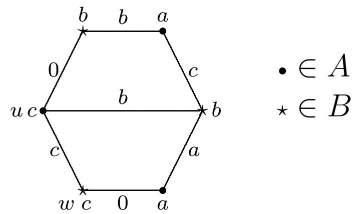

FIGURE 7. Cycle of length 2 mod 4 with chord

the vertices of B with $b \in K$. We shall color the edges of G in such a way that (with a few exceptions) each $v \in A$ with $d_{G'}(v) \geq 2$ will be colored so that $S(v) \cap K \in S_A$ where

$$S_A = \{\ \{0,a\},\ \{a,b\},\ \{b,c\},\ \{0,a,c\},\ \{0,b,c\}\ \},$$

while for $v \in B$, $S(v) \cap K \in S_B$ where

$$S_B = \{\ \{0,b\},\ \{0,c\},\ \{a,c\},\ \{0,a,b\},\ \{a,b,c\}\ \}.$$

This is sufficient, since if $uv \in E(G)$, $u \in A$, $v \in B$, and $S(u) = S(v)$ then $d_{G'}(u) = |S(u) \cap K| = |S(v) \cap K| = d_{G'}(v)$. If this degree is at least 2 then $S(u) \cap K \in S_A$ and $S(v) \cap K \in S_B$ which is a contradiction since $S_A \cap S_B = \emptyset$. However, we can't have $d_{G'}(u) = d_{G'}(v) < 2$ by condition A (if $uv \notin E(G\{1,2\})$) or condition D (if $uv \in E(G\{1,2\})$).

We shall now color each component of G' independently.

Case 1. Good isolated paths.
Using the elements of K, color the edges of a good path arbitrarily so that the sum of the two colors meeting a degree 2 vertex of the path is equal to the color of this vertex. The coloring is avd since if the path has at least 3 vertices and any degree 2 vertex will have $S(v) \cap K \in \{\{0,a\},\{b,c\}\} \subseteq S_A$ if $v \in A$ and $S(v) \cap K \in \{\{0,b\},\{a,c\}\} \subseteq S_B$ if $v \in B$.

Case 2. Good isolated cycles.
If the cycle length is divisible by 4 then we can color the edges from K so that the sum of the two colors meeting a vertex v is equal to the vertex color in K. If the cycle length is not divisible by 4 and there are no color 3 chords then none of the vertices meets color 3 in G. However, by Lemma 3.1 all the neighbors of vertices of the cycle meet all three colors $\{1,2,3\}$ in G. If we give the cycle any avd-coloring using colors from K we are done, since every vertex on the cycle will meet only two colors from $K \cup \{3\}$ whereas their neighbors off the cycle will meet three such colors. (This will be the only case where we do not insist that $S(v) \cap K$ lies in S_A or S_B). Finally, if the cycle has a color 3 chord uv, recolor u and a neighbor w of u on C with color c. Now color the edges around the cycle so that u meets $\{0,c\}$ if $u \in A$ or $\{a,b\}$ if $u \in B$. Then v is still labelled with a or b so the chord uv can be recolored by some color of K making the coloring on C proper (see Figure 7). It is easily checked that $S(v) \cap K$, $S(u) \cap K$, and $S(w) \cap K$ lie in the correct set S_A or S_B as required.

Case 3. Stars of components.
If the central component C_0 is a cycle of length 2 mod 4, relabel one (and only one) vertex $u \in C_0$ that is adjacent to an end-component with b if $u \in A$ and a if $u \in B$. Assume now

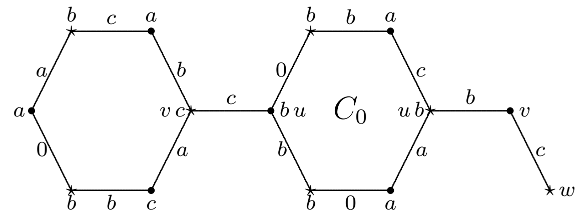

FIGURE 8. Stars of components

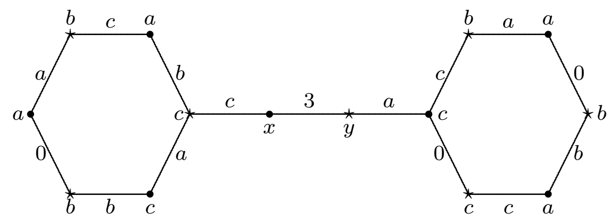

FIGURE 9. Central component is a single edge

that C_0 is not a single edge. Color the central component so that the sum of two colors meeting a degree 2 vertex of C_0 is the vertex color of this vertex. Now recolor the color 3 edges uv from C_0 to C_i ($u \in C_0$, $v \in C_i$) with either 0 or c if $u \in A$, or a or b if $u \in B$. Now for each degree 2 or 3 vertex u of C_0, $S(u) \cap K \in S_A$ if $u \in A$ and $S(u) \cap K \in S_B$ if $u \in B$ (see Figure 8).

Each end-component is bad, so is either a cycle of length 2 mod 4, or a single edge. For each end-component that is a cycle C, let v be the vertex of C joined to the central component C_0. Recolor v and a neighbor of v on C with color $c \in K$. Now color the edges of C so that the colors of K meeting v on C are $\{0, c\}$ if $v \in A$ and $\{a, b\}$ if $v \in B$. Now for each degree 2 or 3 vertex w of C, $S(w) \cap K \in S_A$ if $w \in A$ and $S(w) \cap K \in S_B$ if $w \in S_B$.

For each end-component vw that is a single edge, let uv be the edge joining vw to C_0 (see Figure 8). If $v \in A$ then uv has been colored a or b. For either choice there is a choice of 0 or c on edge vw for which $S(v) \cap K \in S_A$. Similarly, if $v \in B$ then uv has been colored 0 or c. For either choice there is a choice of a or b on edge vw for which $S(v) \cap K \in S_B$.

Finally, assume C_0 is a single edge xy. Then C_0 is joined to two bad components, which by condition A must be cycles (see Figure 9). Recolor the edge xy with color 3. Coloring the edges to the end-components and the edges of the end-components as before we obtain an avd-coloring with x distinguished from y as required.

□

Note that in the proof we only recolored the edges colored 1, 2, or 3 in Lemma 3.1 and for each edge uv either the vertices u and v are distinguished by the colors in $K \cup \{3\}$ or uv is one of the isolated edges of $G\{1, 2, 3\}$ in condition D of Lemma 3.1.

4. General Graphs

The bound in Theorem 1.3 will be obtained by decomposing a general graph into bipartite graphs (Lemma 4.1), and by using an extended version of Lemma 3.1 that makes it possible to color these bipartite graphs 'simultaneously' (Lemma 4.2).

Lemma 4.1. *If G is a k-chromatic graph with no isolated edge or isolated K_3 then G can be written as the edge disjoint union of $\lceil \log_2 k \rceil$ bipartite graphs, each of which has no isolated edge.*

Proof. Let $r = \lceil \log_2 k \rceil$. Then $k \leq 2^r$. We first show that G is the union of r bipartite graphs without the restriction on isolated edges. For $r = 1$ this is clear. For $r > 1$ write $V(G)$ as the union of k independent color classes $V_1, \ldots, V_k$. Partition the classes into two groups $V_1, \ldots, V_{\lceil k/2 \rceil}$ and $V_{\lceil k/2 \rceil + 1}, \ldots, V_k$. Let G_1 be the bipartite graph formed by taking all edges from the first set of color classes to the second. Then $G \setminus E(G_1)$ has chromatic number at most $\lceil k/2 \rceil \leq 2^{r-1}$. Hence, by induction, $G \setminus E(G_1)$ can be written as the edge disjoint union of $r - 1$ bipartite graphs $G_2, \ldots, G_r$. Thus G is the union of r bipartite graphs as required.

Write G as a union of r bipartite graphs in such a way that the total number of isolated edges in the subgraphs G_i is minimized. Suppose there is an isolated edge xy in G_1, say. Since there are no isolated edges in G, there must be some other bipartite graph, G_2 say, with some edge incident to x, say. If we can add xy to G_2 without creating an odd cycle, then by removing xy from G_1 and adding it to G_2 we reduce the number of isolated edges. Hence we may assume there is an even length path from x to y in G_2.

If there are edges zx of G_2 with $d_{G_2}(z) = 1$ then remove all such edges from G_2 and add them to G_1. This gives a star with center x in G_1 and generates no isolated edge in G_2. If no such edge zx exists, remove an edge of an even length path from x to y in G_2 and add it to G_1. Use the edge of this path incident to y if $d_{G_2}(x) > 1$, otherwise use the edge incident to x. This will reduce the total number of isolated edges except in the case when G_2 contains a component consisting of a path xzy of length 2 from x to y.

Since G does not contain an isolated K_3 there must be some other edge meeting $\{x, y, z\}$ in G. Suppose such an edge is incident to either x or y. Then this edge must lie in some other bipartite subgraph, G_3 say. Considering G_3 in place of G_2 we may assume G_3 has a component xwy which is a path of length 2 from x to y. In this case put edge wx in G_1 and wy in G_2. Both G_1 and G_2 remain bipartite and G_3 looses a component. The number of isolated edges in G_1 decreases, contradicting our choice of decomposition into bipartite graphs.

Hence we may assume G has some other edge meeting z but $d_G(x) = d_G(y) = 2$. The edge meeting z lies in G_i where $i = 1$ or $i > 2$. In this case we can move zx to G_i and xy to G_2. Both G_i and G_2 remain bipartite and G_1 looses the isolated edge xy. This reduces the number of isolated edges and contradicts the assumption that there is an isolated edge in some G_j. Hence there is a decomposition into r bipartite graphs each of which has no isolated edge. □

Lemma 4.2. *Assume G is a graph which is the edge-disjoint union of bipartite graphs $G_1, \dots, G_r$, each of which has no isolated edge. Then there exists a proper edge-coloring with colors $\{1_1, \dots, 1_r, 2_1, \dots, 2_r, 3_1, \dots, 3_r, 4, \dots, \chi'(G)\}$ such that colors 1_i, 2_i, and 3_i occur only on the edges of G_i and*

- A *if $uv \in E(G_i) \setminus E(G_i\{1_i, 2_i\})$ then either $\{1_i, 2_i\} \subseteq S(u)$ or $\{1_i, 2_i\} \subseteq S(v)$.*
- B *if C is a cycle in $G\{1_i, 2_i\}$ which does not meet color 3_i in G then $\{1_i, 2_i, 3_i\} \subseteq S(v)$ for every neighbor v in $G_i \setminus C$ of any vertex of C.*
- C *if C is a cycle in $G\{1_i, 2_i\}$ which does meet color 3_i in G, then there exists a $u \in V(C)$ and $uv \in E(G\{3_i\})$ with $\{1_i, 2_i\} \subseteq S(v)$.*
- D *if uv is an isolated edge in $G\{1_i, 2_i, 3_i\}$ then either $S(u) \cap \{4, \dots, \chi'(G)\} \neq S(v) \cap \{4, \dots, \chi'(G)\}$ or there is an edge in G incident to u colored with color 4.*

Proof. By coloring G with $\{1, \dots, \chi'(G)\}$ and splitting colors 1, 2, and 3 into 1_i, 2_i, and 3_i according to which G_i the edge belongs to, we can find a coloring with the given set of colors so that edges colored 1_i, 2_i, or 3_i occur only in G_i. For all such colorings pick one such that

(1) $G\{1_1, \dots, 1_r, 2_1, \dots, 2_r\}$ has maximal edge count,
(2) subject to (1), the sum over i of the number of components of $G\{1_i, 2_i\}$ is minimal,
(3) subject to (1–2), $G\{3_1, \dots, 3_r\}$ has maximal edge count,
(4) subject to (1–3), the number of edges uv failing condition D (for any i) is minimal.

As in the proof of Lemma 3.1 we see that conditions A–C hold for each i. It remains to prove condition D. Let u_1v_1 be an isolated edge of $G\{1_i, 2_i, 3_i\}$. Since G_i contains no isolated edge we can assume $d_{G_i}(u_1) \geq 2$ and u_1 meets another color $k > 4$ on some edge of G_i. Swap colors 4 and k along a Kempe chain (in G) starting at u_1. This will reduce the number of edges failing condition D unless the other end-vertex v_2 of this chain lies in some isolated edge u_2v_2 of $G\{1_j, 2_j, 3_j\}$ and after the recoloring u_2v_2 fails condition D. In this case u_2 also meets color k, so we can form a new Kempe chain starting at u_2 using colors 4 and k. Repeating this process we get a sequence of Kempe chains on colors 4 and k from u_i to v_{i+1}. Eventually this process must terminate with a coloring reducing the number of edges failing condition D, or with some $v_r = v_1$. However in this last case recoloring all these Kempe chains makes both v_1 and u_1 meet color 4. □

Proof of Theorem 1.3.
Since K_3 has a 3-avd-coloring, we can assume G contains no K_3 component. Decompose G using Lemma 4.1 and color G as in Lemma 4.2. Now recolor each bipartite subgraph G_i replacing 1_i, 2_i, 3_i by a set of five colors $K_i = \{0_i, a_i, b_i, c_i, 3_i\}$, disjoint for each i, as in the proof of Theorem 1.2. Some edges uv of G_i may be isolated in $G_i\{1_i, 2_i, 3_i\}$, so u and v will not necessarily be distinguished in G_i, however for all other edges $uv \in E(G_i)$, $S(u) \cap K_i \neq S(v) \cap K_i$ by the comment at the end of Section 3. Also, this recoloring does not change any of the colors $4, \dots, \chi'(G)$ on G. Hence, by condition D, if $uv \in E(G_i)$ and $S(u) \cap K_i = S(v) \cap K_i$ then $4 \in S(u)$. Let H be the subgraph of edges $uv \in E(G)$ such that u and v are not distinguished by the colors in K_i, where $uv \in E(G_i)$. Let H_I be the subgraph of H consisting of all the isolated edges of H. Each vertex in H meets

color 4, so $G\{4\} \cup H_I$ forms a collection of paths and cycles with all edges of H_I on the interior of any path or cycle. Split color 4 into three colors 4_A, 4_B, and 4_C. By alternately changing 4 into 4_A or 4_B along the paths and cycles of $G\{4\} \cup H_I$ we can distinguish the endvertices of each edge of H_I. If a cycle of length 2 mod 4 occurs we shall also need to color one of the color 4 edges of this cycle with 4_C. All other color 4 edges in G may become 4_C without loss of generality. This increases the number of colors used by 2 and distinguishes u and v for all $uv \in E(H_I)$. The graph $H_C = H \setminus H_I$ has no isolated edge and $\Delta(H_C) \leq r \leq \lceil \log_2 k \rceil$. Pick $\chi'_a(H_C)$ new colors and recolor H_C so that it has an avd-coloring using these colors. The resulting coloring is avd. To see this, pick any edge uv of G. If $uv \in E(G_i)$ and $uv \notin E(H)$ then $S(u) \cap K_i \neq S(v) \cap K_i$ since the recoloring of H_C only removes elements from $S(u) \cap K_i$ when u is in an isolated edge of $G_i\{1_i, 2_i, 3_i\}$. But in this case $|S(v) \cap K_i| \geq 2$ (by condition A) and $|S(u) \cap K_i| = 0$. If $uv \in E(H_I)$ then $S(u) \cap \{4_A, 4_B, 4_C\} \neq S(v) \cap \{4_A, 4_B, 4_C\}$ and if $uv \in E(H_C)$ then u and v are distinguished by the $\chi'_a(H_C)$ new colors.

Thus $\chi'_a(G) \leq \chi'(G) - 3 + 5r + 2 + \chi'_a(H_C)$. Finally, $\Delta(H_C) \leq \chi'(H_C) \leq r < \Delta(G)$. So by induction on $\Delta(G)$ we may assume $\chi'_a(H_C) = r + O(\log r)$, and $\chi'_a(G) = \Delta(G) + O(r) = \Delta(G) + O(\log k)$. □

References

[1] M. Aigner, E. Triesch, and Zs. Tuza, Irregular Assignments and Vertex-Distinguishing Edge-Colorings of Graphs, *Combinatorics* **90**, (A. Barlotti et al, eds.). Elsevier Science Pub., New York (1992) 1–9.

[2] P.N. Balister, O.M. Riordan, R.H. Schelp, Vertex-Distinguishing edge colorings of graphs, *Journal of Graph Theory*, to appear.

[3] A.C. Burris and R.H. Schelp, Vertex-Distinguishing Proper Edge-Colourings, *Journal of Graph Theory* **26** (2) (1997) 70–82.

[4] J. Černý, M. Hořňák, and R. Soták, Observability of a Graph, *Math. Slovaca* **46** (1996), 21–31.

[5] O. Favaron, Hao Li, R.H. Schelp, Strong Edge Colorings of Graphs, *Discrete Math.* **159** (1996) 103–109.

[6] Z. Zhongfu, L. Liu, J. Wang, Adjacent strong edge coloring of graphs, *Appl. Math Lett.* **15** (2002) 623–626.

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