Oscillatory Boundary Conditions For Acoustic Wave Equations

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Dedicated to the memory of Philippe Bénilan

1 Introduction

In the textbook literature on theoretical acoustics, it was traditional to use the Robin boundary condition with the wave equation. But it was recognized that this was not the physically correct boundary condition. "Acoustic Boundary Conditions" (or ABC) were introduced in the monograph by Morse and Ingard [13, p.263]. The presentation in [13] is not the usual approach to the wave equations, since the authors treat waves having definite frequency. The time dependent version of ABC was first formulated by Tom Beale and Steve Rosencrans [1] in a very interesting and original paper. ABC will be explained in detail in Section 2.

In the theory of Markov diffusion processes, one studies heat equations of the form

$$\frac{\partial u}{\partial t} = Lu$$

where L is a second order linear elliptic operator (e.g $L = \Delta$). A.D. Wentzell [15] introduced boundary conditions involving second derivatives as well as lower order (Robin type) terms. These parabolic problems were usually studied in spaces of continuous functions. But Favini, Goldstein, Goldstein and Romanelli [8] introduced a new approach to this problem, involving weighted L^p spaces and using the boundary as well as the domain.

For a specific example, consider the heat equation

$$\frac{\partial u}{\partial t} = c^2 \Delta u \quad \text{in } \Omega \Subset \mathbf{R}^n$$

with boundary condition

$$c^2 \Delta u + \beta_1 \frac{\partial u}{\partial n} + \gamma_1 u = 0 \text{ on } \partial\Omega \tag{1.1}$$

where $\beta_1, \gamma_1 \in C(\partial\Omega)$ with $\beta_1 > 0, \gamma_1 \geq 0$ on $\partial\Omega$. The natural L^p space for this problem turns out to be

$$X_p = L^p(\Omega, dx) \oplus L^p\left(\partial\Omega, \frac{c^2}{\beta_1} dS\right), \ 1 \leq p < \infty.$$

On X_2 the semigroup generator G for this problem (i.e. a suitable realization of $c^2\Delta$) is selfadjoint and nonpositive. The corresponding wave equation

$$u_{tt} = Gu$$

is governed by a unitary group on a four component space, based on $H^1(\Omega) \times L^2(\Omega)$ and two copies of $L^2\left(\partial\Omega, \frac{c^2}{\beta_1} dS\right)$. The energy space bears a formal resemblence to the four component energy space that Beale and Rosencrans associated with the wave equation with ABC.

Our goal in this paper is to show that these two versions of the wave equation are closely connected. We shall use these connections to derive new results about both wave equations. Interestingly, the form of the wave equation with ABC most closely associated with the energy conserving wave equation with boundary conditions (1.1) is a nonenergy conserving version; this will be explained in detail in the sequel. Furthermore, a new extension of the boundary condition (1.1) makes the resulting wave equation equivalent to the one with ABC.

The main results are contained in Theorems 1 and 2 (of Section 4) and 4 (of Section 5).

2 Acoustic Boundary Conditions

In this section we explain some of the results of Beale [2]. (The papers [2], [3] expand and develop the work begun in [1].)

Let Ω be a smooth bounded domain in $\mathbf{R}^n, n \geq 1$. (The paper [2] restricts n to be 3. For general n see Gal [11].). Fluid filling Ω is at rest except for acoustic wave motion. Let $\varphi : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ be the velocity potential, so that $-\nabla\varphi(x,t)$ is the particle velocity. Thus φ satisfies the wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \Delta \varphi \quad \text{in } \Omega \times \mathbf{R} \tag{2.1}$$

where $c > 0$ is the (constant) speed of propagation.

Each point x of $\partial\Omega$ is assumed to react to the excess pressure of the acoustic wave like a resistive harmonic oscillator or spring. The normal displacement $\delta(x,t)$ of the boundary into the domain satisfies

$$m(x)\,\delta_{tt}(x,t) + d(x)\,\delta_t(x,t) + k(x)\,\delta(x,t) + \rho\varphi_t(x,t) = 0 \tag{2.2}$$

on $\partial\Omega \times \mathbf{R}$, where ρ is the fluid density and $m, d, k \in C(\partial\Omega)$ with $m > 0, k > 0, d \geq 0$. If the boundary is impenetrable, continuity of the velocity on $\partial\Omega$ implies the compatibility condition

$$\delta_t(x,t) = \frac{\partial\varphi}{\partial n}(x,t) \tag{2.3}$$

on $\partial\Omega \times \mathbf{R}$, where $n = n(x)$ is the unit outer normal to $\partial\Omega$ at x. The energy of the solution is

$$E(t) = \int_\Omega \left(\rho\,|\nabla\varphi|^2 + \frac{\rho}{c^2}\,|\varphi_t|^2\right) dx + \int_{\partial\Omega} \left(k\,|\delta|^2 + m\,|\delta_t|^2\right) dS.$$

Moreover,

$$\frac{dE}{dt} = -2 \int\limits_{\partial\Omega} d\left|\delta_t\right|^2 dS \leq 0,$$

and energy is conserved when the springs are all frictionless, i.e $d \equiv 0$.

The energy (Hilbert) space for this problem is

$$\mathcal{H} = H^1(\Omega) \oplus L^2(\Omega) \oplus L^2(\partial\Omega) \oplus L^2(\partial\Omega). \tag{2.4}$$

Its norm is determined by

$$\|u\|^2 = \int\limits_{\Omega} \left(\rho\left|\nabla\varphi\right|^2 + \frac{\rho}{c^2}\left|\varphi_t\right|^2\right) dx + \int\limits_{\partial\Omega} \left(k\left|\delta\right|^2 + m\left|\delta_t\right|^2\right) dS, \tag{2.5}$$

where $u = (u_1, u_2, u_3, u_4) = (\varphi, \varphi_t, \delta, \delta_t)$. The wave equation with (ABC), (2.1) - (2.3) is equivalent to $u(t) \in \mathcal{D}(A)$ and $u_t = Au$, where

$$A\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ c^2\Delta u_1 \\ u_4 \\ -\frac{1}{m}\left(\rho u_2 + k u_3 + d u_4\right) \end{pmatrix} \tag{2.6}$$

and $\mathcal{D}(A) = \left\{u \in \mathcal{H} : \Delta u_1 \in L^2(\Omega), u_2 \in H^1(\Omega), \frac{\partial u_1}{\partial n} = u_4 \text{ on } \partial\Omega\right\}$. Here in $(Au)_4$, $u_2 \mid_{\partial\Omega}$ makes sense as a member of $H^{\frac{1}{2}}(\partial\Omega)$ (in the trace sense) and $\frac{\partial u_1}{\partial n} = u_4$ is interpreted to mean:

$$\int\limits_{\Omega} \left(\left(\Delta u_1\right)\psi + \nabla u_1 \cdot \nabla\psi\right) dx = \int\limits_{\partial\Omega} \left(u_4\psi\right) dS$$

for all $\psi \in H^1(\Omega)$. The operator A is densely defined in $\mathcal{H}$ and dissipative:

$$\operatorname{Re}\left\langle Au, u\right\rangle = -\int\limits_{\partial\Omega} d\left|u_4\right|^2 dS \leq 0,$$

and A generates a (C_0) contraction semigroup on $\mathcal{H}$, which is a unitary group when $d \equiv 0$. The generator A has interesting spectral properties. Now let $n \geq 3$. (In [2] only $n = 3$ was considered; see Gal [11] for the extension.). Let

$$\sum := \left\{\lambda \in \mathbf{C} : m(x)\lambda^2 + d(x)\lambda + k(x)\lambda = 0 \text{ for some } x \in \partial\Omega\right\}.$$

$\sum$ is compact and symmetric about the real axis. Let R be the unbounded component of $\mathbf{C} \setminus \sum$; note that $0 \in R$. Beale proved that $\lambda \longmapsto (\lambda - A)^{-1}$ is meromorphic on R.

Suppose that m, d and k are constants. Then $\sum$ consists of two points (unless $d^2 = 4mk$), namely

$$\lambda_{\pm} = \frac{1}{2m}\left(-d \pm \sqrt{d^2 - 4mk}\right).$$

Then the essential spectrum of A is $\sum$ and the point spectrum $\sigma_P(A)$ consists of eigenvalue sequences (1) $\{\lambda_n\}$ with $\operatorname{Im}\lambda_n \to \infty, \operatorname{Re}\lambda_n \to 0$, (2) $\{\overline{\lambda_n}\}$, (3) $\{\mu_n^{\pm}\}$ with $\mu_n^{\pm} \to \lambda_{\pm}$, and (4) finitely many additional eigenvalues. In particular, $(\lambda - A)^{-1}$ is not a compact operator when $n \geq 2$. This is remarkable; most linear problems involving the Laplacian on bounded domains have generators with compact resolvents.

The operator A described by (2.6) has the matrix representation

$$A_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ c^2\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & -\frac{\rho}{m}J & -\frac{k}{m}I & -\frac{d}{m}I \end{pmatrix}$$

where I is the identity operator (i.e. $I(u) = u$ for any u) and J means restriction to the boundary: $Ju = u\mid_{\partial\Omega}$. Using the compatibility condition (2.3), we can equally well represent A as the operator matrix

$$A_2 = \begin{pmatrix} 0 & I & 0 & 0 \\ c^2\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ -\frac{d}{m}\frac{\partial}{\partial n} & -\frac{\rho}{m}J & -\frac{k}{m}I & 0 \end{pmatrix}. \tag{2.7}$$

Again, this is just a restatement of (2.6) together with $\frac{\partial u_1}{\partial n} = u_4$. (See the definition of $\mathcal{D}(A)$.)

3 General Wentzell Boundary Conditions

Consider the heat equation

$$\frac{\partial u}{\partial t}=c^2\Delta u \text{ in } \Omega\times[0,\infty)$$

with the general Wentzell boundary condition (or GWBC)

$$\Delta u+\beta\frac{\partial u}{\partial n}+\gamma u=0 \text{ on } \partial\Omega\times[0,\infty),$$

where $\beta,\gamma\in C(\partial\Omega)$ with $\beta>0,\gamma\geq 0$ on $\partial\Omega$. This problem is governed by an analytic contraction semigroup on $X_p, 1\leq p\leq\infty$, where

$$X_p=L^p(\Omega,dx)\oplus L^p\left(\partial\Omega,\frac{1}{\beta}dS\right),\ 1\leq p<\infty \text{ and } X_\infty=C\left(\bar{\bar{\Omega}}\right),$$

with norm, for $u\in C\left(\bar{\bar{\Omega}}\right)\subset X_p$:

$$\|u\|_{X_p}^p=\int_\Omega |u(x)|^p\,dx+\int_{\partial\Omega}|u(x)|^p\,\frac{dS}{\beta(x)},1\leq p<\infty,$$

$$\|u\|_{X_\infty}=\lim_{p\to\infty}\|u\|_{X_p}=\|u\|_{L^\infty(\Omega)}\,.$$

This is proved in [8], except for the analyticity when $p=1,\infty$, which is discussed in [9].

Let G be the generator of this semigroup on X_2 and let G_0 be G restricted to $C^2\left(\bar{\bar{\Omega}}\right)\subset X_2$. Then G_0 is essentially selfadjoint on X_2. This follows as a very special case of the adjoint calculation in [9]. Here we explain it briefly. For $u,v\in\mathcal{D}(G_0)$,

$$\begin{aligned}\frac{1}{c^2}\langle G_0u,v\rangle_{X_2} &= \int_\Omega(\Delta u)\,\bar{v}\,dx+\int_{\partial\Omega}(\Delta u)\,\bar{v}\,\frac{dS}{\beta(x)}\\ &= -\int_\Omega(\nabla u)\cdot(\nabla\,\bar{v})\,dx+\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)\bar{v}\,dS+\int_{\partial\Omega}(\Delta u)\,\bar{v}\,\frac{dS}{\beta}\\ &= \int_\Omega u\,\bar{\bar{\Delta}}v\,dx-\int_{\partial\Omega}u\frac{\partial\,\bar{v}}{\partial n}dS-\int_{\partial\Omega}\gamma u\,\bar{v}\,\frac{dS}{\beta}\end{aligned}$$

since $\Delta u+\beta\frac{\partial u}{\partial n}+\gamma u=0$ on $\partial\Omega$, so

$$\frac{1}{c^2}\langle G_0u,v\rangle_{X_2}=\int_\Omega u\,\bar{\bar{\Delta}}v\,dx+\int_{\partial\Omega}u\,\bar{\bar{\Delta}}v\,\frac{dS}{\beta}=\frac{1}{c^2}\langle u,G_0v\rangle_{X_2}$$

since the GWBC holds for v as well.

So let us consider the wave equation with GWBC:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \text{ in } \Omega \times \mathbf{R}, \tag{3.1}$$

$$\Delta u + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \text{ on } \partial\Omega. \tag{3.2}$$

Note that this boundary condition is identical to (1.1) when $\beta_1 = c^2\beta$ and $\gamma_1 = c^2\gamma$.

As usual, the wave equation $(3.1), (3.2)$ can be written as

$$U_t = \begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix} U = BU$$

(this defines B), where

$$G = c^2 \begin{pmatrix} \Delta & 0 \\ -\beta \frac{\partial}{\partial n} & -\gamma \end{pmatrix};$$

G acts on $X_2 = L^2(\Omega) \oplus L^2\left(\partial\Omega, \frac{dS}{\beta}\right)$ and $\begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix}$ acts on

$$\mathcal{D}((-G)^{\frac{1}{2}}) \oplus X_2 = \widehat{\mathcal{H}}_{en}.$$

The norm in the energy Hilbert space $\widehat{\mathcal{H}}_{en}$ is given on $\mathcal{D}(G_0) \times \mathcal{D}(G_0)$ by

$$\begin{aligned}
\left\| \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \right\|^2_{\widehat{\mathcal{H}}_{en}} &= \left\| (-G)^{\frac{1}{2}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|^2_{X_2} + \left\| \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \right\|^2_{X_2} \qquad (3.3) \\
&= \left\langle (-G) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_{X_2} + \left\langle \begin{pmatrix} w_3 \\ w_4 \end{pmatrix}, \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \right\rangle_{X_2} \\
&= \left\{ c^2 \langle -\Delta w_1, w_1 \rangle_{L^2(\Omega)} + \left\langle \beta \frac{\partial w_1}{\partial n}, w_2 \right\rangle_{L^2\left(\partial\Omega, \frac{dS}{\beta}\right)} + \right. \\
&\quad \left. + \langle \gamma w_2, w_2 \rangle_{L^2\left(\partial\Omega, \frac{dS}{\beta}\right)} + \|w_3\|^2_{L^2(\Omega)} + \|w_4\|^2_{L^2\left(\partial\Omega, \frac{dS}{\beta}\right)} \right\}.
\end{aligned}$$

Here $w_1 = u\mid_\Omega, w_2 = u\mid_{\partial\Omega}, w_3 = \frac{\partial u}{\partial t}\mid_\Omega$ and $w_4 = \frac{\partial u}{\partial t}\mid_{\partial\Omega}$.

Thus

$$B\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ c^2\Delta w_1 \\ -\beta\frac{\partial w_1}{\partial n} - \gamma w_2 \end{pmatrix}. \tag{3.4}$$

Define $\widehat{w} = \begin{pmatrix} \widehat{w_1} \\ \widehat{w_2} \\ \widehat{w_3} \\ \widehat{w_4} \end{pmatrix} = \begin{pmatrix} w_1 \\ w_3 \\ w_2 \\ w_4 \end{pmatrix}$ where $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ is in $\widehat{\mathcal{H}}_{en}$.

This enables us to identify $\widehat{\mathcal{H}}_{en}$ with $\mathcal{H}$. Write $Bw = v$ and define B_1 by $B_1\widehat{w} = \widehat{v}$, i.e. $(Bw)_j = v_j$, for $1 \le j \le 4$ and

$$\widehat{v} = \begin{pmatrix} v_1 \\ v_3 \\ v_2 \\ v_4 \end{pmatrix}.$$

Then

$$B_1\widehat{w} = \begin{pmatrix} w_3 \\ c^2\Delta w_1 \\ w_4 \\ -\beta\frac{\partial w_1}{\partial n} - \gamma w_2 \end{pmatrix} = \begin{pmatrix} \widehat{w_2} \\ c^2\Delta\widehat{w_1} \\ \widehat{w_4} \\ -\beta\frac{\partial \widehat{w_1}}{\partial n} - \gamma\widehat{w_3} \end{pmatrix}.$$

An operator representation of B_1 is

$$B_2 = \begin{pmatrix} 0 & I & 0 & 0 \\ c^2\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ -\beta\frac{\partial}{\partial n} & QJ & -\gamma I & -Q \end{pmatrix} \tag{3.5}$$

where Q is any multiplication operator acting on the boundary and $Ju = u\mid_{\partial\Omega}$, as before. Here B_2 acts on a Hilbert space $\mathcal{H}_{en}$ which is a closed subspace of $\mathcal{H}$, and the components of $\widehat{w} \in \mathcal{D}(B_2)$ are given by $\widehat{w} = \begin{pmatrix} u_1, & u_2, & u_1 \mid_{\partial\Omega}, & u_2 \mid_{\partial\Omega} \end{pmatrix}$. More precisely,

$$\mathcal{H}_{en} = \{u = (u_1, u_2, u_3, u_4) \ \in \ \mathcal{H} : u_3 = u_1 \mid_{\partial\Omega}\}. \tag{3.6}$$

Since u_1 is in $H^1(\Omega)$, it has a trace $u_1 \mid_{\partial\Omega}$ in $H^{\frac{1}{2}}(\partial\Omega)$, and this determines u_3. Note that $\mathcal{H}_{en}$ coincides with a subspace of the space we previously called $\mathcal{H}$, except for the rearrangement of the components of its vectors.

If u is a solution of the wave equation with GWBC, then the fourth component of $B_2\widehat{w}$ is

$$\begin{aligned}
(B_2\widehat{w})_4 &= -\beta\frac{\partial u}{\partial n} \mid_{\partial\Omega} + (Qu) \mid_{\partial\Omega} - \gamma u \mid_{\partial\Omega} - Qu \mid_{\partial\Omega} \\
&= \left(-\beta\frac{\partial u}{\partial n} - \gamma u\right) \mid_{\partial\Omega},
\end{aligned}$$

since Q is a multiplication operator. This a compatibility condition for the problem (analogous to (2.3)).

Note that $\mathcal{H}_{en}$ is a proper closed subspace of $\mathcal{H}$. Let

$$B_3 = B_2 \mid_{\mathcal{H}_{en}} \oplus \ 0 \mid_{\mathcal{H}_{en}^{\perp}}.$$

Then B_3 is densely defined on $\mathcal{H}$ and generates a (C_0) semigroup on $\mathcal{H}$. Moreover, B_3 has the same matrix representation (3.5) as does B_2, except that it acts on a bigger domain.

Rather than compare A and B, which are defined on different spaces, we compare a unitarily equivalent version of A with a unitarily equivalent version of (an extension of) B. This enables us to compare A with B, even though $D(A)$ and $D(B)$ are quite different.

4 Comparing The Boundary Condition

While B_2 is defined on $\mathcal{H}_{en} \subset \mathcal{H}$, (3.5) enables us to extend B_2 and view its extension B_3 as a densely defined operator on $\mathcal{H}$.

We want to make the norm in (3.3) look as much as possible like the norm in (2.4), and make the operator B in (3.4) look as much like the operator A in (2.6) as possible.

The easiest way to do this is to compare the matrix representation A_2 for A in (2.7) and B_3 for B in (3.5) (also on $\mathcal{H}$). So we replace B_1 by B_3 so that B_3 and A_2 are both densely defined operators on $\mathcal{H}$. By making the identifications

$$\beta = \frac{d}{m}, \gamma = \frac{k}{m}, Q = -\frac{\rho}{m}, \tag{4.1}$$

we see that

$$B_3 - A_2 = K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q \end{pmatrix}$$

where $Q = -\frac{\rho}{m}$ is a continuous negative function on $\partial\Omega$. Clearly K has operator norm equal to

$$\|K\| = \left\|\frac{\rho}{m}\right\|_{\infty} = \left\|\frac{\rho}{m}\right\|_{C(\partial\Omega)}$$

and K is compact when $n = 1$ since $L^2(\partial\Omega)$ is 2 dimensional. Finally K is a nonpositive selfadjoint operator on the energy Hilbert space $\mathcal{H}$; the norms are given by

$$\|U\|^2_{\mathcal{H},ABC} = \int_{\Omega} \left(\rho\,|\nabla\varphi|^2 + \frac{\rho}{c^2}\,|\varphi_t|^2\right) dx + \int_{\partial\Omega} \left(k\,|\delta|^2 + m\,|\delta_t|^2\right) dS, \tag{4.2}$$

$$\|U\|^2_{\mathcal{H}_{en},GWBC} = \int_{\Omega} \left(|\nabla u_1|^2 + \frac{1}{c^2}\,|u_2|^2\right) dx + \int_{\partial\Omega} \left(|u_3|^2 + |u_4|^2\right) \frac{dS}{\beta}, \tag{4.3}$$

where we identify $U = (u_1, u_2, u_3, u_4) = (u\,|_{\Omega}, u_t\,|_{\Omega}, u\,|_{\partial\Omega}, u_t\,|_{\partial\Omega})$ for the wave equation with GWBC with

$$U = \left(\sqrt{\rho}\varphi, \sqrt{\rho}\varphi_t, \sqrt{k\frac{d}{m}}\delta, \sqrt{d}\delta_t\right) \tag{4.4}$$

for the wave equation with ABC.

The norm of (4.3) is well defined for $u \in \mathcal{H}$. Our two problems are governed by (C_0) contraction semigroups $S = \{S(t) : t \geq 0\}$ on $\mathcal{H}$ and $T = \{T(t) : t \geq 0\}$ on $\mathcal{H}_{en}$, respectively. Let $\widehat{T}$ on $\mathcal{H}$ be the extension of T described above. The respective infinitesimal generators G_A of S and G_W $(= B_3)$ of $\widehat{T}$ differ by an operator K given in the discussion following (4.1). When we make these identifications, we assume that $d > 0$. We summarize now what the reductions have achieved.

Theorem 1 *Consider the wave equation*

$$u_{tt} = c^2 \Delta u$$

associated with the acoustic boundary condition

$$m\delta_{tt} + d\delta_t + k\delta + \rho\varphi_t = 0, \ \ \delta_t = \frac{\partial \varphi}{\partial n},$$

and the wave equation with general Wentzell boundary condition

$$\Delta u + \beta \frac{\partial u}{\partial n} + \gamma u = 0.$$

Let (4.1) hold. These problems are governed by (C_0) contraction semigroups S on $\mathcal{H}$ and T on $\mathcal{H}_{en}$, respectively; let $\widehat{T}$, with generator $B_3\,(= G_W)$, be the extension of T to $\mathcal{H}_{en}$ described above. Then the generators $G_A\,(= A_2)$ of S and G_W of $\widehat{T}$ differ by an operator which is selfadjoint and bounded on $\mathcal{H}$ and is compact when the dimension of the underlying bounded set $\Omega \subset \mathbf{R}^n$ is one.

By our construction, B_3 is an extension of $G_A + K$, since in extending B_2 to B_3, we got rid of the restriction that $u_3 = u_1 \mid_{\partial\Omega}$ in the domain of B_2. But for λ real and large, $\lambda \in \rho(B_3) \cap \rho(G_A + K)$, and so $B_3 = G_A + K$.

Our construction of B_3 was rather complicated. Associated with A is the compatibility condition (2.3), and this leads to many possible (operator) matrix representations of A. Similarly for B, since there are many choices for the Q in (3.5). Thus our conclusion above is that (a suitable matrix representation of) A is unitarily equivalent to a bounded perturbation of a proper extension of a matrix representation of B.

By semigroup perturbation theory, B_3 generates a (C_0) semigroup $\widehat{T}$ on $\mathcal{H}$ (with norm given by (4.2)) satisfying $\left\|\widehat{T}(t)\right\| \leq e^{\omega t}$, where $\omega \leq \|K\|$, for $t > 0$. Note that, by construction, B_2 generates a (C_0) contraction semigroup T on $\mathcal{H}_{en}$ and that $T(t)$ is the restriction of $\widehat{T}(t)$ to $\mathcal{H}_{en}$; equivalently $\widehat{T}(t)$ is an extension of $T(t)$ to $\mathcal{H}$.

We can make $G_W = G_A$ if we make our wave equation $u_{tt} = c^2 \Delta u$ have the modified GWBC given by

$$\Delta u + \beta \frac{\partial u}{\partial n} + \gamma u + \frac{\rho}{m} \frac{\partial u}{\partial t} = 0 \text{ on } \partial\Omega. \tag{4.5}$$

Let us interpret the GWBC

$$\Delta u + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \text{ on } \partial\Omega$$

as

$$\frac{1}{c^2} u_{tt} + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \text{ on } \partial\Omega$$

where $u_{tt} = c^2 \Delta u$ is assumed to hold on $\bar{\bar{\Omega}}$. It seems natural to add to this a term involving $\frac{\partial u}{\partial t} \mid_{\partial\Omega}$; this is precisely what we did in (4.5). In this case we can identify the δ (in the ABC problem) with a multiple of $u \mid_{\partial\Omega}$ in GWBC problem incorporating (4.5).

When $m = k$ and $d = \rho$, then δ can be identified exactly with the restriction of u to the boundary.

Theorem 2 *Suppose that $m = k$ and $d = \rho$. Then $\mathcal{H}_{en}$ is an invariant subspace of $\mathcal{H}$ for the wave equation with acoustic boundary conditions. Thus the solution at time t satisfies*

$$\varphi(t, \cdot) \mid_{\partial\Omega} = \delta(t, \cdot) \text{ for all } t \geq 0 \text{ if it holds at } t = 0.$$

Thus δ really is φ on the boundary in many cases in the Beale-Rosencrans theory.

5 Compactness Issues

Let A be a closed linear operator on a Banach space with nonempty resolvent set $\rho(A)$; and we let $\mathcal{R}(\lambda, A) = (\lambda I - A)^{-1}$ denote the resolvent operator of A for $\lambda \in \rho(A)$. Then A is called **resolvent compact** if $\mathcal{R}(\lambda, A)$ is compact for some $\lambda \in \rho(A)$ iff $\mathcal{R}(\lambda, A)$ is compact for all $\lambda \in \rho(A)$. The equivalence follows from the resolvent identity

$$\mathcal{R}(\lambda, A) - \mathcal{R}(\mu, A) = (\mu - \lambda) \mathcal{R}(\lambda, A) \mathcal{R}(\mu, A)$$

for all $\lambda, \mu \in \rho(A)$.

Lemma 3 *Let $A_2 = A_1 - P$ where P is a bounded operator. Suppose $\rho(A_1) \cap \rho(A_2) \neq \emptyset$. Then A_2 is resolvent compact iff A_1 is.*

Proof. We can derive the following identity

$$\mathcal{R}(\lambda, A_2) = \mathcal{R}(\lambda, A_1) - \mathcal{R}(\lambda, A_1) P \mathcal{R}(\lambda, A_2) \tag{5.1}$$

for $\lambda \in \rho(A_1) \cap \rho(A_2)$. To prove this simply multiply

$$(\lambda I - A_1) = (\lambda I - A_2) - P$$

on the left by $\mathcal{R}(\lambda, A_1)$ and on the right by $\mathcal{R}(\lambda, A_2)$.

Now suppose $\mathcal{R}(\lambda, A_1)$ is compact. For $\lambda \in \rho(A_2)$ and P bounded, the right hand side of (5.1) (which equals $\mathcal{R}(\lambda, A_2)$) is compact. The equivalence in the lemma now follows by interchanging the indices 1 and 2.□

Let G_{0W} [resp. G_A] be the generator of the (C_0) contraction semigroup governing the wave equation with general Wentzell [resp. acoustic] boundary conditions. Recall that $G_W = B_3$ is the generator of the semigroup $\widehat{T}$ on $\mathcal{H}$ extending $T = \left\{e^{tG_{0W}} : t \geq 0\right\}$. Then by the results of Section 4,

$$G_W - G_A = K$$

where K is a bounded operator. Moreover, K is compact when the dimension n is one and $\rho(G_W) \cap \rho(G_A) \neq \emptyset$ since each resolvent set contains a right half plane.

Theorem 4 *G_W and G_A are both resolvent compact when $n = 1$. Neither G_W nor G_A is resolvent compact when $n \geq 2$.*

Proof. Binding, Brown and Watson [4] – [7] proved that $\Delta = \frac{d^2}{dx^2}$ with GWBC is resolvent compact when $n = 1$. The corresponding eigenvalue problem is

$$u'' = \lambda u \text{ in } \bar{\Omega} = [0, 1],$$

$$\lambda u + (-1)^{j+1} \beta_j u' + \gamma_j u = 0 \text{ at } x = j \in \{0, 1\}.$$

(Recall that $\frac{\partial}{\partial n} = (-1)^{j+1} \frac{d}{dx}$ at $x = j$ for $j = 0, 1$.) Binding, Brown and Watson made a systematic study of such Sturm-Liouville problems with eigenparameter λ in both the equation and the boundary condition. They established an orthonomal basis of eigenvectors in the space

$$\mathcal{H} = L^2(0,1) \oplus \mathbf{C}^2$$

with norm

$$\|u\|_{\mathcal{H}}^2 = \int_0^1 |u(x)|^2 \, dx + \sum_{j=0}^{1} \frac{|u(j)|^2}{\beta_j}$$

and the real eigenvalues tend to $-\infty$. Thus A_W, the $1-$ dimensional Laplacian with GWBC, is selfadjoint and resolvent compact. The corresponding wave equation with the same GWBC is governed by a skewadjoint operator having an orthonormal basis of eigenvectors with eigenvalues $i\mu_n^{\pm}$ with $\mu_n^{\pm}$ real and $\mu_n^{\pm} \to \pm\infty$ as $n \to \infty$.

Thus G_{0W} is resolvent compact. The same conclusion (namely that the operator called G in Section 3 is resolvent compact) was reached independently by Kramar, Mugnolo and Nagel [12] who proved the compactness by a different method on X_p (and not just X_2) when the dimension n is one. In one dimension, the resolvent of G_W is a finite rank extension of the resolvent of G_{0W}; hence it is compact. By Lemma 3, G_A is compact in one dimension.

For $n = 3$, Beale and Rosencrans [1] – [3] showed that G_A is not resolvent compact. In fact, let

$$\sum := \left\{ \lambda \in \mathbf{C} : m(x)\lambda^2 + d(x)\lambda + k(x)\lambda = 0 \text{ for some } x \in \partial\Omega \right\}.$$

Thus G_W has eigenvalue sequences converging to $\pm i\infty$ and to $\sum$. Explicit calculations were given when m, d, k are constants and Ω is a ball, so that $\sum$ consists of two points ω_1, ω_2. Then both ω_1, ω_2 are limit points of eigenvalues of G_A and $\mathcal{R}(\lambda, G_A)$, which is meromorphic on $\mathbf{C} \setminus \sum$, has essential singularities at both ω_1 and ω_2. C. Gal [11] extended the Beale-Rosencrans results to dimension $n \geq 2$.

Independently of Gal's work [11] , using a different method, Delio Mugnolo [14] recently showed that G_A is not resolvent compact in two or more dimensions. Mugnolo dealt with variable coefficients and worked in a very general context involving operator matrices.

So our conclusion is that G_A is not resolvent compact when $n \geq 2$. Neither is G_W by Lemma 3.□

We conjecture that G_{0W} is not resolvent compact in dimension two or more.

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