Complex Analysis Qualifying Exam

Summer 2003

Submit a solution to the first problem, and then choose one of problems 2 or 3.
Unless asked otherwise, please give as much details as you can.

1. True or false (if true, give a short explanation, if false, give a counterexample)
   (a) If an entire function is bounded on the imaginary axis then it is bounded on the entire complex plane.
   (b) If an entire function f satisfies $\lim_{z\to\infty} \frac{f(z)}{z} = 0$ then f is constant.
   (c) If an entire function f satisfies $\lim_{z\to\infty} f(z) = c$, where c is constant then $f(z) = c$ for all $z \in \mathbb{C}$.
   (d) If the sequence (f_n) of entire functions converges uniformly to 0 on $|z| \leq 1$, then, in fact, $f_n(z) \to 0$ for all $z \in \mathbb{C}$.
   (e) Every function analytic on the open unit disk $|z| < 1$ is bounded on $|z| < 1$.

2. (a) State the open mapping theorem for an analytic map from a connected open subset of $\mathbb{C}$ to $\mathbb{C}$.
   (b) State the maximum modulus principle for an analytic map from a connected open subset of $\mathbb{C}$ to $\mathbb{C}$.
   (c) Prove the maximum modulus principle from the open mapping theorem.

3. (a) Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a bounded analytic function. Prove that f is constant.
   (b) Let D be the open unit disk of $\mathbb{C}$ and let $g_1, g_2, \ldots$ and g be analytic functions $D \to \mathbb{C}$. Prove that if the sequence (g_n) converges uniformly to g on every compact subset of D, then (g_n') also converges uniformly to g on every compact subset of D.