Complex Analysis Qualifying Exam Fall 2007

Answer any 3 of the following problems.

1. The Cauchy-Riemann equations in polar form are:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}\,.$$

   (a) Derive them. (You may assume the standard form of the C-R equations.)

   (b) Show that if f is holomorphic on the open set Ω and any one of the real part, the imaginary part, or the modulus of f is constant, then f must be constant.

2. Determine the annulus of convergence of the Laurent series $\sum_{n=-\infty}^{\infty} a^{n^2} z^n$, where $0 < |a| < 1$.

3. Let p, q be polynomials with degree $q >$ (degree p) $+ 1$. If C is a circle containing all of the zeroes of q, show that $\int_C \frac{p(z)}{q(z)} dz = 0$.

4. How many zeroes does $z^7 + 4z^4 + z^3 + 1$ have in $\mathbb{D}$, and also in $\operatorname{ann}(0;\,1,\,2)$?

5. Show that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx \;=\; \frac{\pi}{\sqrt{2}}\,.$$

6. (a) State the Open Mapping Theorem for a holomorphic function from an open, connected subset of $\mathbb{C}$ to $\mathbb{C}$.

   (b) State the Maximum Modulus Principle for a holomorphic function from an open, connected subset of $\mathbb{C}$ to $\mathbb{C}$.

   (c) Prove the latter from the former.