Calculus of Variations Ph.D. Qualifying Exam

April 16, 2005

Instructions. *Solve* ***three*** *of the five problems. Show your work.*

1. Let
$$J[u] = \iiint_{\Omega} \left[\frac{1}{2} \left(u_x^2 + u_y^2 + u_z^2 + u^2 \right) - \rho u \right] \, dx \, dy \, dz.$$
Suppose u_0 is a smooth function that minimizes $J[]$with respect to all smooth functions which vanish on the boundary of Ω. Find the Euler-Lagrange equation for this functional on the region Ω.

2. Find the curves for which the functional
$$J[y] = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y} dx$$
satisfying $y(0) = 0$ can have an extremal if
   (a) The point (x_1, y_1) can vary along the line $y = x - 5$.
   (b) The point (x_1, y_1) can vary along the circle $(x-9)^2 + y^2 = 9$.

3. Find extremals of the functional
$$J[y] = \int_0^1 \left(y'^2 + x^2 \right) \, dx$$
subject to the conditions $y(0) = 0$, $y(1) = 0$ and $\int_0^1 y^2 \, dx = 2$.

4. Suppose a curve $y = y(x)$, $3 \leq x \leq 5$, is used to generate a surface of revolution about the $y-$axis. Find the curve $y = y(x)$ giving the smallest surface area, with $y(3) = 3 \ln 3$, $y(5) = 6 \ln 3$.

5. Write and solve the Hamilton-Jacobi equation corresponding to the functional
$$J[y] = \int_{x_0}^{x_1} y'^2 \, dx,$$
and use the result to describe the extremals of $J[y]$.