

PhD Qualifying Exam: Partial Differential Equations

March 27th 2004

Answer any **three** of the following six questions.

1. Write down an explicit formula for a solution of

$$\begin{aligned}u_t - \Delta u + cu &= f && \text{in } \mathbb{R}^n \times (0, \infty) \\u &= g && \text{on } \mathbb{R}^n \times \{t = 0\},\end{aligned}$$

for $c \in \mathbb{R}$.

2. (a) Show that the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G .

- (b) Using the change of variables $\xi = x + t, \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.
- (c) Use (a) and (b) to derive d'Alembert's formula.

3. Let Ω be a bounded open subset of \mathbb{R}^n with C^1 boundary.

- (a) Define the Hölder spaces $C^{k,\gamma}(\overline{\Omega})$ for $k \in \mathbb{N}$, and $0 < \gamma \leq 1$. What is the norm?
- (b) Define the Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. What is the norm? For which values of p (if any) is $W^{k,p}(\Omega)$ a Hilbert space?
- (c) State the Sobolev Imbedding Theorem for $W^{1,p}(\Omega)$ for both the cases $1 \leq p < n$ and $n > p$.

Turn over

4. Let $u \in C^2(\mathbb{R}^n \times [0, \infty))$ solve the initial value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } \mathbb{R} \times (0, \infty) \\ u &= g, \quad u_t = h && \text{on } \mathbb{R} \times \{t = 0\}. \end{aligned}$$

Suppose g, h have compact support. The kinetic energy is

$$k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) \, dx$$

and the potential energy is

$$p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) \, dx.$$

Prove

- (a) $k(t) + p(t)$ is constant in t ,
 (b) $k(t) = p(t)$ for all large enough times t .
5. Let Ω be a bounded open subset of \mathbb{R}^n with C^1 boundary. Prove the interpolation inequality

$$\int_{\Omega} |Du|^2 \, dx \leq C \left(\int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |D^2u|^2 \, dx \right)^{\frac{1}{2}}$$

for all $u \in C_c^\infty(\Omega)$. (Hint: Use integration by parts.)

6. Assume Ω is connected. Use the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

are $u \equiv \text{constant}$.