PH.D. QUALIFYING EXAM
SPECIAL TOPICS IN BANACH SPACES,
SEPTEMBER 22, 2007

You have 1.5 hour. Solve any two problems. Credit will be given for the best two questions. Show work.

1. Let (Ω, Σ, μ) be a σ-finite measure space and let L^0 be the space of all $f : \Omega \to [-\infty, \infty]$, μ-measurable and finite a.e. on Ω. Define for $f, g \in L^0$,
$$\rho(f,g) = \int_\Omega \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} w(t)\, d\mu(t),$$
where $\int_\Omega w(t)\, d\mu(t) = 1$, and $w(t) > 0$ a.e. on Ω.

   (i) Show that ρ is a metric on L^0.

   (ii) Show that $\{f_n\} \subset L^0$ converges to f in ρ if and only if it converges in measure on every measurable set of finite measure.

2. Let $(X, \|\cdot\|_X)$ be a Banach function space, and $1 \le p < \infty$. Let $X^{(p)} = \{f \in L^0 : |f|^p \in X\}$ and $\|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p}$. Show

   (i) $\|\cdot\|_{X^{(p)}}$ is a norm on $X^{(p)}$.

   (ii) $\|\cdot\|_{X^{(p)}}$ is p-convex.

3. Let (Ω, Σ, μ) be a measure space where μ is a non-atomic infinite measure on Σ. Let L_φ and E_φ be the Orlicz space and its subspace of finite elements over (Ω, Σ, μ), respectively. Prove that $E_\varphi = L_\varphi$ if and only if φ satisfies condition Δ_2.

4. Let $0 < p < \infty$. Prove the following statements.

   (i) The space $L_p[0,1]$ is p-convex and p-concave.

   (ii) The space $L_p[0,1]$ is not r-convex for any $r > p$, and is not r-concave for any $r < p$.