

PhD Qualifying Exam
Graph Theory and Combinatorics

Do **three** of the following seven questions.

1. (a) Prove that the number of partitions of n into at most r terms is equal to the number of partitions of n into (any number of) terms each of which is at most r .
(b) Prove that the number of partitions of n into distinct terms is equal to the number of partitions of n into odd terms.
2. Show that if G is a graph then the following are equivalent.
 - (i) G is 2-connected.
 - (ii) Any 2 vertices of G lie on a common cycle.
 - (iii) Any 2 edges of G lie on a common cycle and G has no isolated points.
3. Let G be a 2-connected graph that is not equal to a cycle. Show that there exists a path P in G with all interior points (if any) of degree 2 in G and such that if the interior edges and vertices of P are removed from G then G remains 2-connected.
4. (a) Show that any r -regular bipartite graph with $r \geq 1$ has a 1-factor.
(b) Show that the edge chromatic number of any bipartite graph is equal to the maximum degree.
5. Let G be a graph on n vertices and let \overline{G} be the edge complement of G . Show that
 - (a) $\chi(G) + \chi(\overline{G}) \leq n + 1$,
 - (b) $\chi(G)\chi(\overline{G}) \geq n$.
6. Let G be an r -regular graph on $2n$ vertices with girth at least g . If $r \geq 2$ and $n \geq 2r^g$, show that there is an $r + 1$ -regular graph G' with girth at least g on the same vertex set as G containing G as a subgraph.
7. Let $G_{n,p}$ be a random graph on n vertices with edges present independently with probability p . For all $\epsilon > 0$ show that
 - (a) if $p \leq (1 - \epsilon) \log n/n$, then $\mathbb{P}(G_{n,p} \text{ is connected}) \rightarrow 0$ as $n \rightarrow \infty$,
 - (b) if $p \geq (1 + \epsilon) \log n/n$, then $\mathbb{P}(G_{n,p} \text{ is connected}) \rightarrow 1$ as $n \rightarrow \infty$.
[You may wish to use the inequality $\binom{n}{r} \leq \left(\frac{en}{r}\right)^r$.]