PhD Qualifying Exam: Topology

March 27th 2004

Do **four** *of the following six problems.*

1. Show that every metric space (X, d) is normal.

2. Let (X, d) be a metric space. Show that X is second countable if and only if X is separable if and only if X is Lindelöf. (Recall a topological space $(X, \mathcal{T})$ is a Lindelöf space if every open cover has a countable subcover.)

3. Let $(X, \mathcal{T})$ be a topological space, let $(Y, \mathcal{U})$ be a Hausdorff space, and let $f\colon X \to Y$ be continuous. Show that the set $\{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is a closed subset of $X \times X$ (in the product topology).

4. Show that the continuous image of a connected set is connected.

5. Show that every compact Hausdorff space is normal.

6. For each $n \in \mathbb{N}$, let $f_n\colon [0, 1] \to \mathbb{R}$ be the function defined by

$$f_n(x) = \begin{cases} nx & \text{if } x \leq \frac{1}{n}; \\ 1 - n(x - \frac{1}{n}) & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n}; \\ 0 & \text{if } \frac{2}{n} \leq x. \end{cases}$$

   Show that for each $x \in [0, 1]$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges. Does the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converge uniformly?