Problem 1: The distribution function of a random variable \( X \) defined on a probability space \((\Omega, \mathcal{F}, P)\) is given by
\[
F(x) = 1 - e^{-x^2} \quad \text{if} \quad x \geq 0, \quad \text{and} \quad F(x) = 0 \quad \text{otherwise}.
\]
(a) Find \( P\{X \geq 2\} \).
(b) Compute the probability density function (with respect to Lebesgue measure \( m \)) of the random variable \( Y = X^{1/3} \).

Problem 2: Let \( X_1 \) and \( X_2 \) be random variables on a probability space \((\Omega, \mathcal{F}, P)\) having Poisson distribution with parameters \( \lambda_1 = 3 \) and \( \lambda_2 = 5 \), respectively. Suppose that \( X_1 \) and \( X_2 \) are independent, that is, for all \( k, j \in \mathbb{N}_0 \),
\[
P\{\{X_1 = k\} \cap \{X_2 = j\} \} := P\{X_1 = k, X_2 = j\} = P\{X_1 = k\}P\{X_2 = j\}.
\]
(1)
Compute \( P\{4 \leq \max\{X_1, X_2\} \leq 10\} \).

Problem 3: Let \( X \) be a random variable on \((\Omega, \mathcal{F}, P)\) with probability density function with respect to Lebesgue measure given by
\[
f(x) = c + dx^2 \quad \text{if} \quad 0 \leq x \leq 1, \quad \text{and} \quad f(x) = 0 \quad \text{otherwise}.
\]
(a) Knowing that \( E\{X\} = \frac{3}{5} \), find \( c \) and \( d \).
(b) Find a formula for the probability density function of the random variable \( Y = \frac{1}{X} \) in terms of \( f \), and use this to compute \( E\left\{\frac{1}{X}\right\} \).

Problem 4: Two random variables \( X \) and \( Y \) on a probability space \((\Omega, \mathcal{F}, P)\) are called uncorrelated if \( E\{XY\} = E\{X\}E\{Y\} \).
(a) If \( X \) and \( Y \) are uncorrelated random variables on \((\Omega, \mathcal{F}, P)\), show that
\[
\mathbb{V}\{X + Y\} = \mathbb{V}\{X\} + \mathbb{V}\{Y\}.
\]
(b) Let \( Z \) and \( W \) be random variables on \((\Omega, \mathcal{F}, P)\) such that \( P\{Z = 0\} = P\{Z = 1\} = P\{Z = 2\} = \frac{1}{3} \), \( P\{W = 0\} = \frac{1}{3} \) and \( P\{W = 1\} = \frac{2}{3} \). Further, assume that \( Z \) and \( W \) are independent (in the sense of definition (1) in Problem 2). Set
\[
X := Z \quad \text{and} \quad Y := Z + W \text{ (mod 3)}.
\]
(i) Show that \( X \) and \( Y \) are uncorrelated, and that they are not independent.
(ii) Compute $V\{X\}$, $V\{7X\}$, $V\{Y\}$ and $V\{X + Y\}$.

Problem 5:

(a) For any $x > 0$ show that
\[ \left( \frac{1}{x} - \frac{1}{x^3} \right) e^{-x^2/2} \leq \int_x^\infty e^{-y^2/2} dy \leq \frac{1}{x} e^{-x^2/2}. \] (3)

(b) Let $X$ be a random variable having standard normal $N(0,1)$ distribution. Use inequalities (3) to estimate $\mathbb{P}\{X \geq 4\}$ and $\mathbb{P}\{1 \leq X \leq 7\}$.

(c) Let $X$ and $Y$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $X$ is $N(0,1)$ distributed, while $Y$ has exponential distribution with parameter $\lambda = 3$. Assume, moreover, that $X$ and $Y$ are uncorrelated. Compute $\mathbb{E}\{(2X + Y)^2\}$ and $V\{X + Y\}$.

Problem 6:

(a) Let $X$ be a nonnegative random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that
\[ \mathbb{E}\{X\} = \int_0^\infty \mathbb{P}\{X > t\} dt = \int_0^\infty \mathbb{P}\{X \geq t\} dt. \] (4)

(b) Suppose that $X$ and $Y$ are nonnegative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying
\[ \mathbb{P}\{Y \geq t\} \leq \frac{1}{t} \int_{y \geq t} X d\mathbb{P}, \quad \forall t > 0. \] (5)

Use formula (4), Fubini’s theorem and Holder’s inequality to prove that for all $c > 1$,
\[ \mathbb{E}\{Y^c\} \leq \left( \frac{c}{c - 1} \right)^c \mathbb{E}\{X^c\}. \] (6)

Problem 7: Let $F : \mathbb{R} \to [0,1]$ be a function satisfying the following conditions:

(i) $F$ is increasing.

(ii) $F$ is right-continuous at every point $x \in \mathbb{R}$.

(iii) $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$

Let $X$ be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X$ is uniformly distributed over $[0,1]$. Set
\[ G(x) := \inf\{y \in \mathbb{R}; F(y) \geq x\}, \quad \forall x \in [0,1]. \]
Show that the distribution function of the random variable $Y := G(X)$ is $F$. Then write down the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ whose distribution function is the given function $F$.

**Problem 8:** Let $\nu$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Use the same type of argument as the one leading to the proof of Theorem 15.24 in [EH] to show that for all $a \in \mathbb{R}$,

$$\nu(\{a\}) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \phi_{\nu}(t) dt.$$  \hspace{1cm} (7)