

**PRACTICE FINAL EXAM**  
**MATH 3242, FALL 05**

Problem 1: Consider the following linear system

$$\begin{aligned}x_1 + x_2 + 3x_3 &= 0 \\x_1 - \frac{1}{2}x_2 + kx_3 &= 0 \\4x_1 + x_2 + 5x_3 &= 0\end{aligned}$$

- a) Determine all values of  $k$  for which the system has a nontrivial solution.
- b) Solve the system in the case  $k = -\frac{1}{2}$ .

Problem 2: Consider the matrices

$$A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -5 & 0 \\ 3 & 0 & 4 \end{bmatrix}.$$

- a) Compute, if possible,  $A^{-1}$  and determine  $\text{Rank}(A)$ .
- b) Compute  $\det(A^T B^3)$  using as few numerical computations as possible and applying properties of matrix operations.

Problem 3: Determine whether each of the following sets of vectors is a subspace of  $\mathbb{R}^4$  and if yes, compute its dimension. Justify.

$$a) H = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : y + w \geq 3x + 2z + 1 \right\}.$$

$$b) W = \left\{ \begin{bmatrix} 2a - 3b \\ -4a + b \\ b \\ a \end{bmatrix} : a, b \text{ scalars} \right\}.$$

Problem 4: Let  $B$  and  $C$  be two  $5 \times 5$  matrices with  $\text{rank}(B) = \text{rank}(C) = 5$ .

- a) If  $A = BC$ , what is  $\text{rank}(A)$ ? Justify.

- b) If  $\det(B) = 7$ , compute  $\det(B^{-1})$ . Justify.
- c) If  $\det(C) = -1$ , compute  $\det(3C)$ . Justify.

Problem 5 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- a) Find  $\dim(\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ .
- b) Let  $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$ , that is, the columns of  $A$  are the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , respectively. Is  $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$  in  $\text{Nul}(A^T)$ ? Justify.

Problem 6:

- a) Suppose that a  $5 \times 6$  matrix  $A$  has four pivot columns. What is  $\dim(\text{Nul}(A))$ ? Justify.
- b) If a  $6 \times 3$  matrix  $A$  has rank 3, find  $\dim(\text{Nul}(A))$ ,  $\dim(\text{Row}(A))$  and  $\text{rank}(A^T)$ . Justify.
- c) Is there a  $6 \times 8$  matrix  $A$  such that  $\dim(\text{Nul}(A)) = 1$ ? If yes, give an example, if no, justify.

Problem 7: Let  $A$  be a  $5 \times 5$  matrix with eigenvalues  $-1, 0, 3$ . Assume that the eigenspace of  $\lambda = 3$  has dimension 3.

- a) Is  $A$  invertible? Justify.
- b) Is  $A$  diagonalizable? Justify.
- c) Is it possible to find 2 independent eigenvectors in the eigenspace corresponding to  $\lambda = -1$ ? If yes, give examples, if no, justify.

d) Suppose that  $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \\ -2 \end{bmatrix}$  is an eigenvector corresponding to the eigenvalue  $\lambda = 3$ .

Compute  $A^4 \vec{v}$ .

Problem 8: Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

- a) Is  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  a spanning set for  $\mathbb{R}^3$ ? Justify.
- b) Compute  $\|\vec{v}_1 - \vec{v}_2\|$  (the distance between  $\vec{v}_1$  and  $\vec{v}_2$ ).
- c) Show that  $\vec{v}_1$  and  $\vec{v}_3$  are orthogonal.

Problem 9: Diagonalize, if possible, the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Problem 10: Find the least squares solution to the system

$$\begin{aligned} -x_1 + x_2 &= 10 \\ 2x_1 + x_2 &= 5 \\ x_1 - 2x_2 &= 20. \end{aligned}$$