**Math 312 EXAM 2** (90 minutes) NAME: ______________________

There are 120 points on this exam. You need only 100 to get a perfect score.
Show **ALL** work. Answers without work and/or explanations will receive no credit.
GOOD LUCK!

1. True/False with proofs. If the statement is true, explain briefly why. If the statement is false, give a counterexample.

**(a)** (5 points) If both $\lim_{n\to\infty} s_{2n}$ and $\lim_{n\to\infty} s_{2n+1}$ exist then $\lim_{n\to\infty} s_n$ also exists.

**(b)** (5 points) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.

**(c)** (5 points) If $f(\frac{1}{n}) = (-1)^n$ for every positive integer $n$ then $f$ is discontinuous at the point 0.

**(d)** (5 points) If $f$ is a continuous function and $f(0) = -f(1)$ then $f(x_0) = 0$ for some $x_0 \in [0, 1]$.

2. **(a)** (5 points) Define what is a subsequence and a subsequential limit point of a sequence $(s_n)$.

**(b)** (15 points) Prove that $+\infty$ is a subsequential limit point of a sequence $(s_n)$ if and only if $(s_n)$ is not bounded from above.

3. **(a)** (10 points) Does the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$$

converges? Does it converges absolutely?

**(b)** (10 points) Decide whether the following series converge

$$\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n^3+1}}, \qquad \sum_{n=1}^{\infty} \frac{3^n n!}{n^n}.$$

(In case you need it $\lim_{n\to\infty}(1+1/n)^n = e < 3$.)

4. **(a)** (10 points) If $g$ is continuous at the point $x_0$ and $f$ is continuous at the point $g(x_0)$ then prove that $h(x) = f(g(x))$ is continuous at the point $x_0$.

**(b)** (10 points) Find the points of continuity of the functions $f$ and $g$ defined as follows:

(i) $f(x) = x^2 \cos(1/x)$ for $x \neq 0$ and $f(0) = 0$,

(ii) $g(x) = \cos(1/x)$ for $x \neq 0$ and $g(0) = 0$.

5. **(a)** (10 points) If $f$ is continuous on the interval $[a, b]$ then prove that $f$ is bounded on $[a, b]$.

**(b)** (10 points) Suppose that $f$ is given by the formula

$$f(x) = \frac{e^{\cos x}}{x^2 + 1}.$$

Prove that $f$ has a positive minimum on the interval $[0, 100]$, that is, there exists $x_0 \in [0, 100]$ such that $f(x) \geq f(x_0) > 0$ for every $x \in [0, 100]$.

6. **(a)** (10 points) Prove that there exists an $x_0 \in (-\pi/2, \pi/2)$ such that

$$\frac{\cos x_0}{x_0} = 1.$$

**(b)** (10 points) Suppose that $f$ and $g$ are continuous functions such that $f(x) \neq g(x)$ for every $x$. Show that either $f(x) > g(x)$ for every $x$, or $f(x) < g(x)$ for every $x$.