Assignment 2- Due February 24

(**1**) Let $(X, \mathcal{B}, \mu, T)$ be an *ergodic* measure preserving system and f be a measurable function such that $\int_X f^- d\mu < +\infty$ and $\int_X f^+ d\mu = +\infty$. Prove that for a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = +\infty.$$

(**2**) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, $f \in L^1(\mu)$, and $a \in \mathbb{R}$. Prove that for a.e. $x \in X$ the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n a} f(T^n x)$$

converge as $N \to \infty$. (Actually we can choose the set of measure zero where convergence possibly fails to be independent of a but this is harder to prove.)

(**3**) (i) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and $A \in \mathcal{B}$ with $\mu(A) > 0$. Show that for a.e. $x \in A$ the set of n's for which $T^n x \in A$ has positive limiting frequency, that means,

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N \colon T^n x \in A\}|}{N} > 0.$$

(ii) Is it true that for a.e. $x \in A$ the set of n's for which $T^n x \in A$ has bounded gaps?

(**4**) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, and let $\mathcal{S} \subset \mathcal{B}$ be such that for every $A \in \mathcal{B}$ and $\varepsilon > 0$ there exists $B \in \mathcal{S}$ with $\mu(A \triangle B) < \varepsilon$.
(i) If

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n} B) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{S}$, prove that the system is ergodic.

(ii) Can we conclude ergodicity if we just assume that the limit in (i) is positive for all $A, B \in \mathcal{S}$?

(**5**) Show that for (Lebesgue) a.e. $x \in [0, 1)$ with continued fraction expansion $x = [a_1, a_2, \ldots]$ the limit

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}}$$

exists. Compute the limit.