

Assignment 3- Due March 25

(1) Prove that T is weakly mixing or strongly mixing if and only if T^k or $T^{1/k}$ is for some $k \geq 1$. (By $T^{1/k}$ we mean any measure preserving transformation S , if there exists one, such that $S^k = T$.)

(2) Let (X, \mathcal{B}, μ, T) be a measure preserving system. Then:

(i) T is weakly mixing if and only if for every integer sequence $(n_k)_{k \in \mathbb{N}}$ with positive density, and $A \in \mathcal{B}$ with $\mu(A) > 0$, we have that $\mu(\bigcup_{k=1}^{\infty} T^{-n_k} A) = 1$.

(ii) If T is strongly mixing then for every increasing integer sequence $(n_k)_{k \in \mathbb{N}}$, and $A \in \mathcal{B}$ with $\mu(A) > 0$, we have that $\mu(\bigcup_{k=1}^{\infty} T^{-n_k} A) = 1$.

(3) (i) Show that a measure preserving system (X, \mathcal{B}, μ, T) is weakly mixing if and only if for all sets $A, B, C \in \mathcal{B}$ with positive measure there exists $n \geq 1$ such that

$$\mu(A \cap T^{-n} B \cap T^{-2n} C) > 0. \quad (1)$$

(Note that one direction was "almost" proved in class.)

(ii) Is the same true if we replace (1) with

$$\mu(A \cap T^{-n} B) \mu(A \cap T^{-n} C) > 0?$$

(4) (i) If (X, \mathcal{B}, μ, T) is a weakly mixing system show that for every $a \in (0, 1)$ and $f \in L^1(\mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i n a} f(T^n x) = 0 \quad (2)$$

for μ -a.e. $x \in X$.

(ii) Suppose that for some measure preserving system (2) holds for some $f \in L^2(\mu)$ and $a \in [0, 1)$. Is it true that f is orthogonal to every $e^{2\pi i a}$ -eigenfunction?

(5) We say that an integer sequence $(a_n)_{n \in \mathbb{N}}$ is *good for L^2 -convergence* if for every measure preserving system (X, \mathcal{B}, μ, T) and $f \in L^2(\mu)$ the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^{a_n} x)$$

converge in L^2 as $N \rightarrow \infty$.

(i) For $f \in L^2$ and $M, N \in \mathbb{N}$ show that

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f(T^{a_n} x) - \frac{1}{M} \sum_{n=0}^{M-1} f(T^{a_n} x) \right\|_{L^2(\mu)} = \left\| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i a_n t} - \frac{1}{M} \sum_{n=0}^{M-1} e^{2\pi i a_n t} \right\|_{L^2(\sigma_f)},$$

where σ_f is the spectral measure of f .

(ii) Show that the sequence $(a_n)_{n \in \mathbb{N}}$ is good for L^2 -convergence if and only if the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i a_n t}$$

exists for every $t \in [0, 1)$. Conclude the Von Neumann ergodic theorem.