ERGODIC RAMSEY THEORY

Assignment 2- Due: February 28

(**1**) Let $\alpha \in \mathbb{R}$ and $f\colon \mathbb{T} \to \mathbb{R}$ be measurable. Define $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x,y) = (x+\alpha, y+f(x)) \pmod 1.$$

Prove that the measure preserving system $(\mathbb{T}^2, \mathcal{B}, \mu, T)$ (μ is the Haar measure on $\mathbb{T}^2$) is ergodic if and only if α is irrational *and* for every $m \in \mathbb{N}$ the functional equation

$$mf(x) = h(x+\alpha) - h(x) \pmod 1$$

has no measurable solution $h\colon \mathbb{T} \to \mathbb{R}$.

(**2**) For $x \in [0,1)$ let $a_1(x), a_2(x), \ldots$ denote the coefficients of the continued fraction expansion of x. Show that for (Lebesgue) a.e. $x \in [0,1)$ the harmonic mean of its coefficients, i.e. the limit

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1(x)} + \ldots + \frac{1}{a_n(x)}},$$

exists. Express the limit as a series series.

(**3**) (*i*) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{B}$ with $\mu(A) > 0$. Show that for a.e. $x \in A$ the set of n's for which $T^n x \in A$ has positive limiting frequency, i.e.

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N \colon T^n x \in A\}|}{N} > 0.$$

(Notice that this is a strengthening of Problem 3 in HW 1.)

(*ii*) Is it true that for a.e. $x \in A$ the set of return times

$$\Lambda_x = \{n \in \mathbb{N} \colon T^n x \in A\}$$

has bounded gaps? (Compare your answer with the statement of Problem 2 in HW 1.)

(**4**) We say that an integer sequence $(a_n)_{n\in\mathbb{N}}$ is *good for L^2-convergence* if for *every* measure preserving system $(X, \mathcal{B}, \mu, T)$ and $f \in L^2(\mu)$ the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x)$$

converge in L^2 as $N \to \infty$.

(*i*) For $f \in L^2$ and $M, N \in \mathbb{N}$ show that

$$\Big\|\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x) - \frac{1}{M}\sum_{n=1}^{M} f(T^{a_n}x)\Big\|_{L^2(\mu)} = \Big\|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i a_n t} - \frac{1}{M}\sum_{n=1}^{M} e^{2\pi i a_n t}\Big\|_{L^2(\sigma_f)},$$

where σ_f is the spectral measure of f.

(*ii*) Show that an integer sequence $(a_n)_{n\in\mathbb{N}}$ is good for L^2-convergence if and only if the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i a_n t}$$

exists for *every* $t \in [0, 1)$.

(*iii*) Conclude that if p is a polynomial with integer coefficients then the sequence $(p(n))_{n\in\mathbb{N}}$ is good for L^2-convergence.

(**5**) Follow the argument we used to prove the theorem of Furstenberg and Sárközy to prove the following result of Kamae & Mendès-France:
"Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive integers that satisfies:
(*i*) The sequence $(r_n\alpha)_{n\in\mathbb{N}}$ is uniformly distributed in $\mathbb{T}$ for every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.
(*ii*) The set $\{n \in \mathbb{N}\colon d|r_n\}$ has positive density for every $d \in \mathbb{N}$.
Then $R = \{r_1, r_2, \ldots\}$ is a set of recurrence (or equivalently, an intersective set)."[1]

[1]Notice that this result easily implies the Furstenberg-Sárközy theorem. Moreover, using classical results about the prime numbers one can also conclude that the set of shifted primes $\{p - 1\colon p \text{ prime}\}$ is a set of recurrence (that's another result of Sárközy). It is probably true, but not proven yet, that the set of shifted primes is actually good for k-recurrence for every $k \in \mathbb{N}$.

(**6**) (*i*) Let α be irrational. Show that for every measure preserving system $(X, \mathcal{B}, \mu, T)$ and $f \in L^2(\mu)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(e^{2\pi i n^2 \alpha} \int \bar{f} \cdot T^n f \; d\mu \Big) = 0.$$

(*ii*) Use part (*i*) to show that if α is irrational then the set

$$R = \big\{ n \in \mathbb{N} \colon \{n^2\alpha\} \in [1/2, 3/4] \big\}$$

is a set of recurrence.

(*iii*) Show that the set $R^2 = \{r^2 \colon r \in R\}$ is *not* a set of recurrence and R is *not* a set of 2-recurrence (see HW 1 for the definitions).[2]

Remark: For Problems 4 and 6 feel free to use the following fact (we are going to prove it later): If p is a polynomial with at least one non-constant coefficient irrational then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i p(n)} = 0.$$

[2] Using a somewhat more complicated argument it is possible to show that the set $R = \big\{ n \in \mathbb{N} \colon \{n^{k+1}\alpha\} \in [1/2, 3/4] \big\}$ is a set of k-recurrence but *not* a set of $(k+1)$-recurrence. Also, similar ideas can be used to show the following: Given any set $G \subset \mathbb{N}$ there exists a set R such that: $R^g = \{r^g \colon r \in R\}$ is a set of recurrence if and only if $g \in G$.