Assignment 3- Due April 1

(**1**) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $k \in \mathbb{N}$. Prove that $T^k f = f$ if and only if f is a linear combination of λ-eigenfunctions, where $\lambda^k = 1$.

(**2**) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Then T is weakly mixing if and only if for every integer sequence (n_k) with positive density, and $A \in \mathcal{B}$ with $\mu(A) > 0$, we have that $\mu(\cup_{k=1}^{\infty} T^{-n_k} A) = 1$.

(**3**) Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing measure preserving system. Show that:
(i) For every $A, B \in \mathcal{B}$ we have

$$\text{D-}\lim_{n\to\infty} \mu(A \cap T^{-n^2} B) = \mu(A)\mu(B).$$

(ii) (Harder) For every $A, B, C \in \mathcal{B}$ we have

$$\text{D-}\lim_{n\to\infty} \mu(A \cap T^{-n} B \cap T^{-n^2} C) = \mu(A)\mu(B)\mu(C).^{1}$$

(**4**) We say that a set $\Lambda \subset \mathbb{Z}$ of positive density is *weak mixing*, if the measure preserving system constructed using the correspondence principle of Furstenberg is weak mixing. Show that if Λ is a weak mixing set, then the equations $x + y = z$ and $y^2 + z = w$ are solvable within Λ. Is the system $x + y = z,\ y^2 + z = w$ solvable within Λ? [2]

[1] A more general result involving $k+1$ sets and k polynomials with pairwise non-constant differences was proved by V. Bergelson.

[2] Related results can be found in the thesis of A. Fish (http://arxiv.org/abs/0711.3050).

(**5**) For every $k \in \mathbb{N}$, show that there exist an ergodic measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$, such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) < \frac{1}{2}\,\mu(A)^k.$$

Hint: For every $\varepsilon > 0$, we know that there exists $N \in \mathbb{N}$, and a subset Λ of $\{1, \ldots, N\}$, such that $|\Lambda| \geq N^{1-\varepsilon}$ and Λ does not contain non-trivial 3-term arithmetic progressions. If ε is small enough, use this to construct a periodic system and a set with the advertised property.

(**6**) Show that the following two systems are isomorphic:
(i) The nilsystem $(X = G/\Gamma, T_a, m)$ where $G = \mathbb{R}^3$ is the Heisenberg group,[3] $\Gamma = \mathbb{Z}^3$, m is the Haar measure on X, and $a = (\alpha, \beta, \gamma)$.
(ii) The skew product system $(\mathbb{T}^2, S, \lambda)$,where λ is the Haar measure on $\mathbb{T}^2$, and $S\colon \mathbb{T}^3 \to \mathbb{T}^3$ is defined by

$$S(x, y) = (x + \alpha, y + \beta, z + f(x, y)),$$

where $f\colon \mathbb{T}^2 \to \mathbb{T}$ is equal to

$$f(x, y) = (x + \alpha)[y + \beta] - x[y] - \alpha y + \gamma.$$

Remark: It can be shown that this system is NOT isomorphic to any system induced by an affine transformation on some finite dimensional torus.

[3]Multiplication is defined by $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$,