Assignment 4- Due April 28

(**1**) Let X be a compact metric space and $T\colon X \to X$ be a minimal homeomorphism of X. Show that for every $x \in X$, and every nonempty open subset V of X, the set $\{n \in \mathbb{Z}\colon T^n x \in V\}$ has bounded gaps.

(**2**) Use the unique ergodicity of some irrational rotation to prove that for almost every $n \in \mathbb{N}$ the 10-expansion of the number 2^n contains the digit 7.[1]

(**3**) Let $a(n)$ be a sequence of integers such that for every *ergodic* measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-a(n)} A) > 0.$$

Show that the sequence $a(n)$ is good for single recurrence.

(**4**) Pick $n \in \mathbb{N}$ independently with probability $\sigma_n = 1/\log n$. Show that, almost surely, the resulting set of integers contains arbitrarily large intervals, and consequently, it is good for multiple recurrence.[2]

(**5**) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and assume that T^2 is ergodic.

(i) If $f, g, h \in L^\infty$ prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f(x) g(T^n x) h(T^{2n} x) d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f(x) g(T^{2n} x) h(T^{4n} x) d\mu.$$

(This identity can be generalized to include more terms but the proof becomes much harder.)

[1]It is an open problem whether the same holds for all but finitely many n's.

[2]Bourgain has shown that whenever $n\sigma_n \to \infty$, then almost surely, the resulting set is good for single recurrence. It is an open problem whether the same holds for double or higher order recurrence. On the other hand, it is known that if $\limsup n\sigma_n < \infty$ then, almost surely, the resulting set is *not* good for single recurrence.

(*ii*) If $A \in \mathcal{B}$ and $\mu(A) > 0$, show that there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-(2n+1)}A \cap T^{-2(2n+1)}A) > 0$. (Unfortunately, no nice combinatorial implication of this fact is known.)

(**6**) (*i*) Let α be irrational and $k \in \mathbb{N}$. Show that for every measure preserving system $(X, \mathcal{B}, \mu, T)$ and $f \in L^\infty(\mu)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(e^{2\pi i n^{k+1}\alpha} \int f \cdot T^n f \cdot \ldots \cdot T^{kn} f \; d\mu \right) = 0.$$

(*ii*) Use part (*i*) and Furstenberg's multiple recurrence theorem to show that if α is irrational, and $k \in \mathbb{N}$, then the set

$$R = \left\{ n \in \mathbb{N} \colon \{n^{k+1}\alpha\} \in [1/2, 3/4] \right\}$$

is a set of k-recurrence. (An argument similar to the one used in HW 2, shows that R is *not* good for $(k+1)$-recurrence.)