7261 Homework Assignment 3 Fall 2006

Due Friday, October 13.

1. Define the generalized quaternion group Q_m by

$$Q_m = \left\langle a, b \mid a^{2m} = 1,\ b^2 = a^m,\ ab = ba^{-1} \right\rangle .$$

   Show that $|Q_m| = 4m$. [Hint: Consider the subgroup of matrices generated by $\begin{pmatrix} e^{\pi i/m} & 0 \\ 0 & e^{-\pi i/m} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.]

2. If G is a finite p-group and $1 \neq H \trianglelefteq G$, show that $H \cap Z(G) \neq 1$

3. If P is a p-Sylow subgroup of G, show that $N_G(N_G(P)) = N_G(P)$.

4. Show there are no simple groups of orders 104, 176, 182, or 312.

5. Let G be a group of order pq where $p > q$ and p and q are both primes. Show that G is either isomorphic to the cyclic group of order pq or is generated by two elements a and b with $a^p = b^q = 1$ and $bab^{-1} = a^r$ where $r^q \equiv 1 \bmod p$ and $r \not\equiv 1 \bmod p$. Show further that this second case occurs only when $p \equiv 1 \bmod q$, and that in this case any two choices of r give rise to isomorphic groups, so there are up to isomorphism only two groups of order pq. [Note: If $q = 2$ this gives the dihedral group D_p. Hint: Recall there exists an integer g mod p such that every integer $1,\ldots,p-1$ is congruent mod p to a power of g.]