7281 Homework Assignment 4 Fall 2007

Due Tuesday, November 20th.

Let A be an $n \times n$ matrix. Define a *minimal polynomial* of A to be a non-zero polynomial $m(x)$ of smallest degree such that $m(A) = 0$. (Since multiplying $m(x)$ by a non-zero constant does not affect the definition, we may assume the leading term of $m(x)$ has coefficient 1.)

1. (a) Show that if $h(x)$ is a polynomial with $h(A) = 0$ then $h(x) = m(x)q(x)$ for some polynomial $q(x)$.

   (b) Deduce that $m(x)$ is unique (up to multiplication by a constant).

   (c) If $A^2 = A$, what are the possibilities for $m(x)$?

2. (a) Show that if $m(\lambda) = 0$ then λ is an eigenvalue of A.
   [Hint: Use Cayley-Hamilton and Question 1.]

   (b) Show that if λ is an eigenvalue of A then $m(\lambda) = 0$.
   [Hint: apply $m(A)$ to an eigenvector of A.]

3. Recall that a matrix is nilpotent if $A^N = 0$ for some N. What is the characteristic polynomial of a nilpotent matrix?

4. (a) Show that the minimal polynomial of A and $P^{-1}AP$ are the same.

   (b) Show that if A is diagonalizable then the minimal polynomial has distinct roots.

   (c) Show that if A has distinct roots then A is diagonalizable. [Hint: it is enough to show that every vector is a linear combination of eigenvectors.]

5. (a) Find the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

   (b) For each of the eigenvalues in (a), find a basis for the corresponding eigenspace, together with the geometric and algebraic multiplicities of the eigenvalue.