7361 Homework 1 Solutions Fall 2005

1. Find the three solutions to the equation $z^3 - 3z + 1 = 0$.
   [Hint: Use the substitution $z = w + 1/w$.]

   Substituting $z = w+1/w$ in the equation gives $w^3+1+w^{-3} = 0$, or $(w^3)^2+(w^3)+1 = 0$. The solutions to this quadratic are $w^3 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Writing in terms of polar coordinates $w^3 = \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}$. Hence $w = \cos\theta \pm i\sin\theta$ where $3\theta = \frac{2\pi}{3} + 2\pi n$. Now $1/w = \cos(-\theta) \pm i\sin(-\theta) = \cos\theta \mp i\sin\theta$, so $z = w + 1/w = 2\cos\theta$. Three possible values of θ are $\frac{2\pi}{9}$, $\frac{8\pi}{9}$, and $\frac{14\pi}{9}$ giving solutions $z = 2\cos\frac{2\pi}{9}$, $2\cos\frac{8\pi}{9}$, and $2\cos\frac{14\pi}{9} = 2\cos\frac{4\pi}{9}$.

2. Define the *cross-ratio* of four complex numbers (z_1, z_2, z_3, z_4) to be

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

   (a) Show that the cross-ratio is invariant under Möbius transformations, i.e., if $w_i = \frac{az_i+b}{cz_i+d}$ then

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

   This can be done by direct substitution, but an easier method is to note that if $w_i = 1/z_i$ then $\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(1/z_1-1/z_2)(1/z_3-1/z_4)}{(1/z_1-1/z_4)(1/z_3-1/z_2)} = \frac{z_1z_2z_3z_4(z_2-z_1)(z_4-z_3)}{z_1z_2z_3z_4(z_4-z_1)(z_2-z_3)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$, while if $w_i = az_i + b$ then $\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(az_1-az_2)(az_3-az_4)}{(az_1-az_4)(az_3-az_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$. Now any Möbius transformation is a composition of transformations of these forms since if $w = \frac{az+b}{cz+d}$ and $c \neq 0$ then $w = \frac{1}{c}(bc - ad)u + \frac{a}{c}$, where $u = 1/v$ and $v = cz + d$. Hence $\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

   (b) Find a Möbius transformation which sends 1, 2, i to 2, 1, $2i$ respectively.
   [Hint: The cross-ratio of $(2, 1, 2i, w)$ equals the cross-ratio of $(1, 2, i, z)$.]

$$\frac{(2-1)(2i-w)}{(2-w)(2i-1)} = \frac{(1-2)(i-z)}{(1-z)(i-2)}$$

$$\Rightarrow \frac{2i-w}{2-w} = \frac{-(2i-1)(i-z)}{(i-2)(1-z)} = \frac{(2i-1)z+(2+i)}{(2-i)z+(i-2)}$$

$$\Rightarrow \frac{2i-2}{2-w} = \frac{2i-w}{2-w} - 1 = \frac{(3i-3)z+4}{(2-i)z+(i-2)}$$

$$\Rightarrow 2 - w = \frac{(2i-2)((2-i)z+(i-2))}{(3i-3)z+4} = \frac{(6i-2)z+(2-6i)}{(3i-3)z+4}$$

$$\Rightarrow w = 2 - \frac{(6i-2)z+(2-6i)}{(3i-3)z+4} = \frac{-4z+(6+6i)}{(3i-3)z+4}.$$

3. Write $f(z) = u(x,y) + iv(x,y)$ and $z = x + iy = re^{i\theta}$.

(a) Show that in polar coordinates the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Write $x = r\cos\theta$, $y = r\sin\theta$, so $f(z) = u(r\cos\theta, r\sin\theta) + iv(r\cos\theta, r\sin\theta)$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta,$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta).$$

Hence $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta,$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta).$$

Hence $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

(b) Show that the function $\log z = \log r + i\theta$ is holomorphic in the region $r > 0$, $-\pi < \theta < \pi$.

$$\frac{\partial \log r}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial \theta}{\partial \theta}$$

$$\frac{\partial \theta}{\partial r} = 0 = -\frac{1}{r}\frac{\partial \log r}{\partial \theta}.$$

The partial derivatives are all continuous, so $\log z$ is holomorphic.

4. If $f(z)$ is differentiable, show that $g(z) = \overline{f(\overline{z})}$ is also differentiable. What is the derivative of g (in terms of f')?

$g(z+h) - g(h) = \overline{f(\overline{z}+\overline{h}) - f(\overline{z})} = \overline{\overline{h}f'(\overline{z}) + |h|\epsilon(h)} = h\overline{f'(\overline{z})} + |h|\overline{\epsilon(h)}$ where $\epsilon(h)$ (and hence $\overline{\epsilon(h)}$) tends to zero as $\overline{h}$ (and hence h) tends to 0.

Alternatively: write $f(x+iy) = u(x,y) + iv(x,y)$. Then, writing subscripts for partial derivatives, $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$. Now $g(x+iy) = u(x,-y) - iv(x,-y)$. Thus $\frac{\partial}{\partial x}\mathrm{re}(g) = u_x(x,-y) = v_y(x,-y) = \frac{\partial}{\partial y}\mathrm{im}(g)$. Also $\frac{\partial}{\partial y}\mathrm{re}(g) = -u_y(x,-y) = v_x(x,-y) = -\frac{\partial}{\partial y}\mathrm{im}(g)$. Thus Cauchy-Riemann equations hold for g. Hence g is differentiable (here we use the fact that f' is continuous, but this holds since in fact f is infinitely differentiable). Also $g'(x+iy) = u_x(x,-y) - iv_x(x,-y) = \overline{f'(x-iy)} = \overline{f'(\overline{z})}$.

5. Show that if $f\colon S \to \mathbb{C}$ is a continuous function and S is path connected then the image of f, $f(S) = \{f(z) : z \in S\}$, is also path connected.

Pick any two points $w_1, w_2 \in f(S)$. Then $w_1 = f(z_1)$, $w_2 = f(z_2)$ for some $z_1, z_2 \in S$. Since S is path connected, there is a path $\gamma\colon [a,b] \to \mathbb{C}$ such that $\gamma(a) = z_1$, $\gamma(b) = z_2$. But then $f(\gamma(t))$ is a composition of continuous functions, so is a continuous function of t. It always takes values in $f(S)$, and $f(\gamma(a)) = w_1$, $f(\gamma(b)) = w_2$. Thus $f(\gamma(t))$ is a path from w_1 to w_2 inside $f(S)$. Since this holds for any w_1 and w_2, $f(S)$ is path connected.