7361 Homework 2 Solutions Fall 2005

1. Let $\gamma\colon [0,1] \to \mathbb{C}$ be the path given by $\gamma(t) = t + it^2 \sin(1/t^2)$, $\gamma(0) = 0$.

   (a) Show that γ is continuously differentiable on $(0,1]$, and has a (one-sided) derivative at $t = 0$.

$\gamma'(t) = 1 + 2it\sin(1/t^2) - \frac{2i}{t}\cos(1/t^2)$ exists and is continuous for $t > 0$. At $t = 0$, $\lim_{h\to 0^+} \frac{\gamma(h)-\gamma(0)}{h} = \lim_{h\to 0^+}(1 + ih\sin(1/h^2)) = 1$, so the one-sided derivative exists at 0.

   (b) Show that γ has infinite length.

If $t_n = 1/\sqrt{\pi(n+.5)}$ then $\mathrm{im}\gamma(t_n) = \pm\frac{1}{\pi(n+.5)}$ with sign $+$ if n is even and $-$ is n is odd. Thus $|\gamma(t_n) - \gamma(t_{n+1})| \geq \frac{1}{\pi(n+.5)}$. Hence the length of γ is at least $\sum_n |\gamma(t_n) - \gamma(t_{n+1})| \geq \sum_n \frac{1}{\pi(n+.5)} = \infty$.

[This shows that we need smooth curves not only to have a derivative at the endpoints, but that the derivative must also be continuous at the endpoints.]

2. Show that

$$\sin(2\pi/7) + \sin(4\pi/7) + \sin(8\pi/7) = \sqrt{7}/2.$$

[Hint: Let $\omega = e^{2\pi i/7}$ and consider $a+b$ and ab where $a = \omega+\omega^2+\omega^4$ and $b = \omega^3+\omega^5+\omega^6$.] $a + b = \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = (\omega - \omega^7)/(1 - \omega)$. But $\omega^7 = e^{2\pi i} = 1$, so $a+b = (\omega-1)/(1-\omega) = -1$. Similarly $ab = \omega^4+\omega^6+\omega^7+\omega^5+\omega^7+\omega^8+\omega^7+\omega^9+\omega^{10} = 3+(\omega^4+\omega^6+\omega^5+\omega^1+\omega^2+\omega^3) = 3-1 = 2$. Now $a = 2/b$, so $a+2/a = -1$, and $a^2+a+2 = 0$. Solving this quadratic gives $a = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$. Since $\omega^n = \cos(2\pi n/7) + i\sin(2\pi n/7)$, the sum of sin's is the imaginary part of a. Thus this sum is $\pm\sqrt{7}/2$. A simple estimation shows that it is also positive (e.g., $\sin(4\pi/7) > 0$ and $\sin(2\pi/7)+\sin(8\pi/7) = \sin(2\pi/7)-\sin(\pi/7) > 0$ since $\sin(\pi/7) < \sin(2\pi/7)$).

3. Define the Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, ...) by $a_0 = a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

   (a) Show that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a strictly positive radius of convergence R. [Hint: show $a_n \leq 2^n$.]

By induction on n, $a_0 = 1 \leq 2^0$, $a_1 = 1 \leq 2^1$. For $n \geq 2$, $a_n = a_{n-1}+a_{n-2} \leq 2^{n-1}+2^{n-2} = 3.2^{n-2} \leq 2^n$. Thus $a_n \leq 2^n$ for all $n \geq 0$ and $1/R = \limsup |a_n|^{1/n} \leq \limsup |2^n|^{1/n} = 2$. Hence $R \geq \frac{1}{2} > 0$.

   (b) Show that $f(z) = \frac{1}{1-z-z^2}$ for $|z| < R$.

$(1 - z - z^2)\sum a_n z^n = (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots) - (a_0 z + a_1 z^2 + a_2 z^3 + \dots) - (a_0 z^2 + a_1 z^3 + \dots) = a_0 + (a_1 - a_0)z + (a_2 - a_1 - a_0)z^2 + \cdots + (a_n - a_{n-1} - a_{n-2})z^n + \cdots = 1 + (1-1)z + 0z^2 + \cdots + 0z^n + \cdots = 1$. This calculation is valid inside the minimum of the radius of convergence of $1 - z - z^2$ (infinite) and the radius of convergence of $\sum a_n z^n$ $(= R)$. Thus $f(z) = 1/(1 - z - z^2)$ for $|z| < R$.

   (c) By writing $\frac{1}{1-z-z^2} = \frac{a}{1-\alpha z} + \frac{b}{1-\beta z}$ for suitable constants a, b, α, β, find an explicit expression for a_n.

$\frac{a}{1-\alpha z}+\frac{b}{1-\beta z}=\frac{(a+b)-(a\beta+b\alpha)z}{(1-\alpha z)(1-\beta z)}$ so we need $(1-\alpha z)(1-\beta z)=1-z-z^2$, $a+b=1$, $a\beta+b\alpha=0$. Factoring the quadratic gives $\alpha,\beta=\frac{1\pm\sqrt{5}}{2}$. Say $\alpha=\frac{1+\sqrt{5}}{2}$, $\beta=\frac{1-\sqrt{5}}{2}$, then $a(\alpha-\beta)=\alpha$, so $a=\alpha/\sqrt{5}$, $b=1-a=-\beta/\sqrt{5}$. Now $a/(1-\alpha z)=\sum_{n=0}^{\infty}a\alpha^n z^n$ for $|z|<1/|\alpha|$, and $b/(1-\beta z)=\sum_{n=0}^{\infty}b\beta^n z^n$ for $|z|<1/|\beta|$. Thus $\sum a_n z^n=\sum(a\alpha^n+b\beta^n)z^n$ if $|z|<\min(R,1/|\alpha|,1/|\beta|)$. By uniqueness of the Taylor series

$$a_n=a\alpha^n+b\beta^n=\frac{1}{\sqrt{5}}\left(\left(\tfrac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\tfrac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

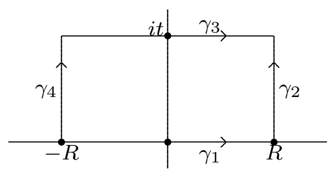
(d) What is the radius of convergence R?

Clearly converges for $|z|<\min(1/|\alpha|,1/|\beta|)=1/\alpha$, but does not converge for $z=1/\alpha$, so $R=1/\alpha=\frac{\sqrt{5}-1}{2}$.

4. Assume $t>0$ is real. By integrating $e^{-z^2/2}$ around the rectangle with vertices $\pm R$, $\pm R+it$, and then letting $R\to\infty$ show that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}\cos(tx)\,dx=e^{-t^2/2}$$

[You may use the fact that $\int_{-\infty}^{\infty}e^{-x^2/2}=\sqrt{2\pi}$.]



Let

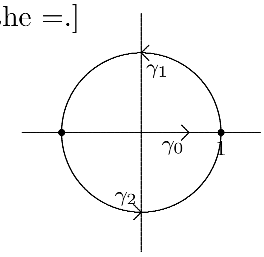
$\gamma_1(s)=s,\quad -R\le s\le R;$
$\gamma_2(s)=R+is,\quad 0\le s\le t;$
$\gamma_3(s)=s+it,\quad -R\le s\le R;$
$\gamma_4(s)=-R+is,\quad 0\le s\le t.$

Then since $e^{-z^2/2}$ is analytic everywhere, $\int_{\gamma_1+\gamma_2-\gamma_3-\gamma_4}e^{-z^2/2}dz=0$. Now $\int_{\gamma_1}e^{-z^2/2}dz=\int_{-R}^{R}e^{-s^2/2}ds\to\sqrt{2\pi}$ as $R\to\infty$. Also if $z=\pm R+is$, $0\le s\le t$, then $|e^{-z^2/2}|=|e^{-R^2/2\mp Rsi+s^2/2}|=e^{-R^2/2+s^2/2}\le e^{-R^2/2+t^2/2}$. Thus $|\int_{\gamma_2}|,|\int_{\gamma_4}|\le te^{-R^2/2+t^2/2}\to 0$ as $R\to\infty$. Finally $\int_{\gamma_3}=\int_{-R}^{R}e^{-s^2/2-ist+t^2/2}ds=e^{t^2/2}\int_{R}^{R}e^{-x^2/2}e^{-itx}dx$ which must tend to $\sqrt{2\pi}+0+0$ as $R\to\infty$. Thus $\int_{-\infty}^{\infty}e^{-x^2/2}e^{-itx}dx=\sqrt{2\pi}e^{-t^2/2}$. Taking real parts and dividing by $\sqrt{2\pi}$ gives the result.

5. Let $f(z)=c_0+c_1z+\cdots+c_nz^n$ be a polynomial. Show that

$$\left|\int_{-1}^{1}f(x)^2\,dx\right|\le\frac{1}{2}\int_{0}^{2\pi}|f(e^{i\theta})|^2d\theta=\pi\sum_{k=0}^{n}|c_k|^2.$$

[Use Cauchy's Theorem (twice) for the $\le$, and direct calculation for the $=$.]



Let

$\gamma_0(t)=t,\quad -1\le t\le 1;$
$\gamma_1(t)=e^{it},\quad 0\le t\le\pi;$
$\gamma_2(t)=e^{it},\quad \pi\le t\le 2\pi.$

Then $\gamma_0+\gamma_1$ and $\gamma_0-\gamma_2$ are closed curves.
Hence $\int_{-1}^{1}f(x)^2\,dx=\int_{\gamma_0}f(z)^2\,dz=-\int_{\gamma_1}f(z)^2\,dz=\int_{\gamma_2}f(z)^2\,dz$.
Now $|\int_{\gamma_1}f(z)^2\,dz|=|\int_0^{\pi}f(e^{it})^2ie^{it}dt|\le\int_0^{\pi}|f(e^{it})|^2dt$.
Similarly $|\int_{\gamma_2}f(z)^2\,dz|\le\int_{\pi}^{2\pi}|f(e^{it})|^2dt$.
Thus $|\int_{-1}^{1}f(x)^2\,dx|=\frac{1}{2}|\int_{\gamma_1}f(z)^2\,dz|+\frac{1}{2}|\int_{\gamma_2}f(z)^2\,dz|\le\frac{1}{2}\int_0^{2\pi}|f(e^{it})|^2dt$.
Now $|f(e^it)|^2=(\sum_k c_ke^{ikt})\overline{(\sum_j c_je^{ijt})}=\sum_{j,k}c_k\overline{c_j}e^{i(k-j)t}$. But $\int_0^{2\pi}e^{int}=\frac{1}{in}e^{int}|_0^{2\pi}=0$ if $n\ne 0$ and is 2π if $n=0$, so $\frac{1}{2}\int_0^{2\pi}|f(e^{it})|^2=\frac{1}{2}\sum_k 2\pi c_k\overline{c_k}=\pi\sum_k|c_k|^2$.