7361 Homework Assignment 3 Fall 2005

Answers due Thursday, October 13.

1. Consider $f(x+iy) = \frac{(x+iy)xy^2}{x^2+y^4}$, $x+iy \neq 0$, $f(0) = 0$.

   (a) Show that $\frac{f(z)-f(0)}{z-0}$ tends to 0 as $z \to 0$ along any straight line (i.e., $z = \alpha t$ for some constant $\alpha \in \mathbb{C}$ and real $t \to 0$).

   (b) Show that $f'(0)$ does not exists. [This shows that in computing f' it is not sufficient to consider the limit along straight lines.]

2. Suppose $f(z) = \sum_n a_n z^n$ is holomorphic on $D = \{z : |z| < 1\}$ and $|f(z)| \leq \frac{1}{1-|z|}$. Show that

$$|a_n| \leq (n+1)(1+\tfrac{1}{n})^n < (n+1)e.$$

3. Find all functions that are holomorphic on $D = \{z : |z| < 1\}$ and for which

$$f\left(\tfrac{1}{n+1}\right) = \tfrac{1}{n^2+n+1} \qquad n = 1, 2, 3, \ldots$$

4. Suppose $f(z)$ is holomorphic on $D = \{z : |z| < 1\}$ and $|f(\frac{1}{n+1})| < 2^{-n}$, $n = 1, 2, 3, \ldots$ Show that $f(z) = 0$ on D.

5. Assume $f(z)$ is non-zero and analytic on the whole complex plane. Suppose $|f'(z)| \leq C|f(z)|$ for some constant $C > 0$. Show that $f(z) = \mu e^{\lambda z}$ for some constants λ, μ.