Solutions submitted by members of the Cantor Sect are denoted with a *.

Problem: Prove the striking identity

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k} = \frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}$$

***Solution 1:**

$$\begin{aligned}
\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k} &= \int_0^1 \sum_{k=0}^{n}(-1)^k x^{2k}\binom{n}{k} dx \\
&= \int_0^1 (1-x^2)^n dx \\
&= \int_0^1 (1+x)^n(1-x)^n dx \\
&= \int_1^2 u^n(2-u)^n du && (u = 1+x) \\
&= \frac{1}{2}\int_0^2 u^n(2-u)^n du && \text{(symmetry)} \\
&= 4^n \int_0^1 t^n(1-t)^n dt && (t = \tfrac{u}{2}) \\
&= \frac{4^n (n!)^2}{(2n+1)!} && \text{(Beta Function)} \\
&= \frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}
\end{aligned}$$

Remark: This is also a good problem to use the following fact:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k) = g(n) \iff \sum_{k=0}^{n}(-1)^k\binom{n}{k}g(k) = f(n)$$

***Solution 2:** Let S_n denote the desired sum.

$$\begin{aligned}
S_n = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k} &= \sum_{k=0}^{n} \frac{(-1)^k n}{(2k+1)(n-k)}\binom{n-1}{k} \\
&= \frac{n}{2n+1}\sum_{k=0}^{n}(-1)^k \left(\frac{1}{n-k} + \frac{2}{2k+1}\right)\binom{n-1}{k} \\
&= \frac{2n}{2n+1}S_{n-1} + \frac{1}{2n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k} \\
&= \frac{2n}{2n+1}S_{n-1} \\
&= \frac{(2n)\cdots 4\cdot 2}{(2n+1)\cdots 5\cdot 3} \qquad \text{(Iteration)}
\end{aligned}$$

***Solution 3:** Make the whole sum the coefficient of x^{2n} in a power series in order to use Fubini's Theorem.

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{n}{k}\right)x^{2n} \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{n}{k}\right)x^{2n} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\frac{x^{2k}}{(1-x^2)^{k+1}} \\
&= \frac{1}{2x\sqrt{x^2-1}}\sum_{k=0}^{\infty}\frac{1}{k+1/2}\left(\frac{x^2}{x^2-1}\right)^{k+1/2} \\
&= \frac{1}{2x\sqrt{x^2-1}}\left(\ln\left(x-\sqrt{x^2-1}\right)-\ln\left(x+\sqrt{x^2-1}\right)+i\pi\right) \\
&= \frac{1}{2x\sqrt{x^2-1}}\left(2\ln\left(x-\sqrt{x^2-1}\right)+i\pi\right)
\end{aligned}$$

where the last step uses the fact the lucky fact that

$$\frac{1}{x+\sqrt{x^2-1}} = x-\sqrt{x^2-1}$$

Now lets evaluate

$$G(x) = 1+\sum_{n=1}^{\infty}\frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}x^{2n}$$

Note that $G(x)$ satisfies the differential equation

$$G(x) = \frac{1}{x^2}\int\frac{(xG(x))'-1}{x}dx, \qquad G(0)=1$$

which simplifies to the first order linear differential equation

$$G'(x)+\frac{2x^2-1}{x^3-x}G(x) = -1, \qquad G(0)=1$$

with solution

$$G(x) = \frac{1}{2x\sqrt{x^2-1}}\left(2\ln\left(x-\sqrt{x^2-1}\right)+i\pi\right)$$

Thus $F(x)=G(x)$ which gives the desired result.